Implementation of switching law constrained for controlling of switched linear systems

Aplicación de la ley de conmutación restringida para el control de sistemas lineales conmutados
Implementação de lei de comutação restrita para controle de sistemas lineares comutado

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Abstract

A special class of switched linear systems with switching law constrained to logical state-input can be employed to model a wide range of different systems. The present paper presents a new stability analysis and controller design method for this class of hybrid systems. Proposed methods is based on the quadratic Lyapunov function. Stability analysis and design of these systems have resulted in solving a convex optimization problem of Linear Matrix Inequality type. The results of simulation on dc-dc buck converter confirm the effectiveness of proposed method.

Keywords: Switched Linear Systems, Constrained Switching Law, Quadratic Lyapunov Function, Linear Matrix Inequality.

Resumen

Se puede emplear una clase especial de sistemas lineales conmutados con ley de conmutación restringida a entrada de estado lógico para modelar una amplia gama de sistemas diferentes. El presente documento presenta un nuevo método de análisis de estabilidad y diseño de controlador para esta clase de sistemas híbridos. Los métodos propuestos se basan en la función cuadrática de Lyapunov. El análisis de estabilidad y el diseño de estos sistemas han dado como resultado la solución de un problema de optimización convexo de tipo de desigualdad de matriz lineal. Los resultados de la simulación en el convertidor dc-dc buck confirman la efectividad del método propuesto.

Palabras claves: sistemas lineales conmutados, ley de conmutación restringida, función de Lyapunov cuadrática, desigualdad de matriz lineal.

Resumo

Uma classe especial de sistemas lineares comutados com lei de comutação restrita a entrada de estado lógico pode ser empregada para modelar uma ampla gama de diferentes sistemas. O presente artigo apresenta um novo método de análise de estabilidade e design de controlador para esta classe de sistemas híbridos. Os métodos propostos são baseados na função quadrática de Lyapunov. A análise de estabilidade e o projeto desses sistemas resultaram na solução de um problema de otimização convexa do tipo Desigualdade de Matriz Linear. Os resultados da simulação no conversor dc-dc buck confirmam a eficácia do método proposto.

Palavras-chave: sistemas lineares comutados, lei de comutação restrita, função quadrática de lyapunov, desigualdade de matriz linear.

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Introduction

Hybrid systems are a specific category of systems that are temporally continuous or discrete. They contain discrete state events and are employed to model a broad range of systems (Rodrigues, 2002).

A specific class of hybrid systems is the switched linear systems with constrained switching law. This constraint can be present in the minimum time intervals between switching instances (dwell time) (Sen, & Ibeas, 2008) (Shorten et all, 2007), switching rate [3], switching based on state variables [4] and switching sequence. The specific class investigated here is the switched linear systems with switching law constrained to logical state-input. Power electronics converters like Buk and Boost in discontinuous working condition and series resonance converter are examples of these systems. Although stability analysis and controller design of the considered class are of great importance, there have been no specific literature examining this issue.

Two known classes of hybrid systems are piecewise affine system (PWA) (Rodrigues, 2002) (Rantzer & Johansson, 2000) (Johansson, 2002) and switched linear system (SLS) without constraint on switching law (Sun & Shuzhi, 2004).From now on, wherever the word SLS is used, it means switched linear system without any constraints on switching law. PAW system includes affine subsystems and switching law constrained to state variables. The investigated system in this paper has a switching law created by combining switching laws of PWA and SLS systems. The stabilization analysis of PWA systems is typically conducted by multiple Lyapunov functions (MLF). The implemented approach here is such that each subsystem or state-space region is associated with a pseudo Lyapunov function. The main problem of employing MLF is to find the appropriate pseudo Lyapunov function. The quadratic Lyapunov function is a right candidate that its stability conditions have been expressed based on Linear Matrix Inequality (LMI) in (Johansson, 2002) (Pettersson & Lennartson, 2002). Piecewise quadratic stability of these systems has been evaluated in Ref [4]. SLS stabilization is another important issue for which various methods have been represented [9]. The quadratic stabilization has been explained with the help of minimum trajectory method. This method is usable for non-linear systems.

Regarding the modelling capability in MLD form (Bemporad & Morari, 1999), there is a possibility to convert the form of investigated class to MLD form to do the analysis and design. However, there have not been any considerable research work on this field. Model Predictive Control (MPC) has been utilized to control MLD systems (Beccuti et all, 2009). This controller possesses high computational complexity and is hard to be implemented. A computational solution based on data tables has been proposed in Ref (Bemporad et all, 2002) to implement real-time of this controller approach. Buk and Boost converters were modeled and controlled by MLD model in discontinuous conduction mode in Refs (Hejri & Mokhtari, 2009) (Hejri & Mokhtari, 2010) Converting the equations of converters to MLD form entail computational complexity and is hard to be controlled. The method proposed in this paper, without converting the system form, solves the stability analysis and controller design with the assistance of quadratic Lyapunov function.

Main innovations of this paper are as follows: 1. presenting some theorems to analyze stability and design stable switching law for switched linear system constrained to logical state-input, 2. stable analysis and controller design of the switching law for the investigated system is converted to a convex optimization problem in LMI form.

The considered class in this paper is introduced in section 2 where its model is also expressed. A review of PAW systems and stabilization techniques along with their control is carried out in section 3. In section 4, the proposed method is introduced for stabilization and control. Then in section 5, stable analysis of the proposed approach is presented. Simulation results are then illustrated in section 6. Finally, concluding remarks of the paper are given in section 7.

Class of Switched Linear Systems with Switching Law Constrained to Logical State-Input

Here in this section, we give a mathematical description of this type of systems. In general, hybrid systems are expressed as follows [16]:

$$\dot{x}(t) = f(x(t), i(t), u(t))$$
$$y(t) = g(x(t), i(t), u(t)) \quad (1)$$
$$i(t^+) = f(x(t), i(t), u(t), S(t))$$

Where $S(t)$, $u(t)$, $i(t)$, and $x(t)$ are output variable, logical input vector, continuous input, subsystem number, and system states, respectively. Here, $S = \begin{bmatrix} s_1 & s_2 & \dots & s_{M_s} \end{bmatrix}^T$, $l = 1,2, \dots, M_s$, $s_1 \in \{0,1\}$, $i \in \{1,2, \dots, M\}$, $y(t) \in R$, and $x(t) \in R^n$. M_s and M are number of logical inputs and subsystems, respectively. Typically, two types of switching law can occur in hybrid systems, namely arbitrary and constrained. A specific type of constrained switching is on the basis of state variables and is mathematically described as follows:

$$x \in \{x | H_j^T x - g_i < 0\} \Leftrightarrow \varphi(x) = j \quad (2)$$

Where j is the number of active subsystem, n is the number of state variables, p_j is the number of descriptive boundaries of jth mesh, $H_j \in R^{p_j \times n}$, $g_i \in R^{p_j \times 1}$, and $j = 1,2, \dots, M$. "Mesh" in this paper refers to a convex region in state space that is described by relation (2). Similarly, switching constrained to the system states refers to a constraint in the form of relation (2).

The system investigated in this paper can be expressed as:

$$\dot{x}(t) = A_i x(t) + b_i$$
$$y(t) = C_i x(t) + d_i \quad (3)$$
$$i(t^+) = \varphi(i(t), S(t), x(t))$$

Where x, y , and i $(\le i \le M)$ are state variable, outlet variable, and number of active subsystem, respectively. S is logical input variable, n is the number of state variable, $d_i \in R$, $C_i \in R^{1 \times n}$, $b_i \in R^{n \times 1}$, and $A_i \in R^{n \times n}$. Furthermore, the switching law (φ) is specified, given by:

$$\varphi(i(t), S(t), x(t)) = i:$$
$$H_g^T x - g_i < 0 \;\&\; S = \begin{bmatrix} s_1 & s_2 \dots s_{M_s} \end{bmatrix}^T \quad (4)$$

Where M_s is the number of logical inputs $(1 \le k \le M_s)$, s_k is the value of kth logical input, p_j is the number of descriptive boundaries of jth mesh, $1 \le j \le M_{mesh}$, $H_j \in R^{p_j \times n}$, and $g_i \in R^{p_j \times 1}$. Active subsystem is determined based on the location of state variables and values of logical inputs. Here, the control input is the same logical input. Total number of subsystems is also determined by relation (5).

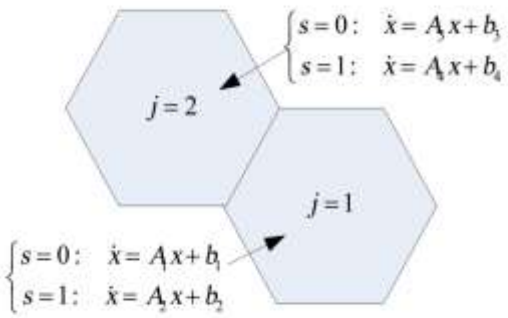

Figure 1. Graphical description of investigated systems (in particular, $M_{mesh} = 2$ and $M_s = 1$)

Piecewise Quadratic Stability of PWA Systems

Typical variables used to define the PWA system along with theoretical prerequisites are represented in this section. A PWA system is described as [4]:

$$\dot{x} = A_i x + B_i u + b_i, x \in R_i, i = 1,2, \dots, M \quad (6)$$

Where M is the number of meshes and i-th mesh is described by R_i. R_i meshes are a subset of χ, ($R_i \subseteq \chi \subseteq R^n$), and partition it. Additionally, stability of controllers class, that are in the form of $u = K_i x + m_i$, is investigated within the R_i mesh. Thus, a closed-loop system is then obtained as:

$$\dot{x} = (A_i + B_i K_i)x + B_i m_i + b_i \square \overline{A_i}x + \overline{b_i} \quad (7)$$

Meshes are considered as convex polyhedrons, thus; it can be said that each R_i mesh is generated by the intersection of p_i and the hyper-plane in R^n space. It can be illustrated as follows:

$$\begin{aligned} R_i &= \{H_i^T x - g_i < 0\} \\ H_i &= [h_{i1} \;\; h_{i2} \;\; \dots \; h_{ip_i}] \\ g_i &= [g_{i1} \;\; g_{i2} \;\; \dots \; g_{ip_i}]^T \end{aligned} \quad (8)$$

In order to analyze the stability, Lyapunov function is used in the form of:

$$\begin{aligned} V(x) &= \sum_{i=1}^{M} \beta_i(x) V_i(x), V(x) > 0 \\ V_i(x) &= x^T P_i x + 2q_i^T x + r_i \end{aligned} \quad (9)$$

Where P_i is symmetric and positive definite, $i = 1,2, \dots, M$, and $\beta_i(x) = \begin{cases} 1, & x \in R_i \\ 0, & otherwise \end{cases}$. V_i in relation (9) can be rewritten as:

$$V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad (10)$$

Time derivative of $V_i(x)$ is expressed as follows:

$$\frac{d}{dt} V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \overline{A_i^T} P_i + P_i \overline{A_i} & P_i \overline{b_i} + \overline{A_i^T} q_i \\ \overline{b_i^T} q_i + q_i^T \overline{A_i} & 2\overline{b_i^T} q_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad (11)$$

Lyapunov function (9) has a switching rate of α_i in each mesh and for $\varepsilon \geq 0$:

$$x \in R_i \Rightarrow \begin{cases} V_i(x) > \varepsilon \left\| x - x_{cl}^i \right\|_2 \\ \frac{d}{dt} V_i(x) < -\alpha_i V_i(x) \end{cases} \quad (12)$$

Conditions of relation (12) are substantially conservative and lead to positivity of Lyapunov function and negativity of it derivative throughout entire space for each i -index. In the following, conditions are more conveniently expressed so as to reduce the conservatively. With the help of relations (8)-(9) and S-procedure [18, 19, and 20], Lyapunov function with a convergence rate of $\alpha_i$in R_i mesh is positive provided that $P_i > 0$

and arrays of q_i, r_i, Z_i, and Λ_i are all positive (Rodrigues & How, 2003).In other words:

$$\begin{bmatrix} P_i - \varepsilon I_n - \overline{H_i^T} Z_i \overline{H_i} & q_i + \varepsilon x_{cl}^i + \overline{H_i^T} Z_i \overline{g_i} \\ q_i^T + \varepsilon x_{cl}^{i\,T} + \overline{g_i^T} Z_i \overline{H_i} & r_i - \varepsilon x_{cl}^{i\,T} x_{cl}^i - \overline{g_i^T} Z_i \overline{g_i} \end{bmatrix} > 0 \qquad (13)$$

Lyapunov function is negative if [17]:

$$\begin{bmatrix} \left\{\overline{A_i^T} P_i + P_i \overline{A_i} + \overline{H_i^T} \Lambda_i \overline{H_i} + \alpha_i P_i\right\} & \left\{P_i \overline{b_i^T} + \overline{A_i} q_i - \overline{H_i^T} \Lambda_i \overline{g_i} + \alpha_i q_i\right\} \\ \left\{\overline{b_i^T} P_i + q_i^T \overline{A_i} - \overline{g_i^T} \Lambda_i \overline{H_i} + \alpha_i q_i^T\right\} & \left\{2\overline{A_i^T} q_i + \overline{g_i^T} \Lambda_i \overline{g_i} + \alpha_i r_i\right\} \end{bmatrix} < 0 \qquad (14)$$

Where $\overline{H_i} = \begin{bmatrix} 0 & h_{i1} \; h_{i2} \ldots \; h_{ip_i} \end{bmatrix}^T$, $\overline{g_i} = \begin{bmatrix} 0 & g_{i1} \; g_{i2} \ldots \; g_{ip_i} \end{bmatrix}^T$, and I_n is identity matrix. The theorem for controller design in Ref (Rodrigues & How, 2003)has resulted in resolving a series of binary matrix inequality.

Proposed Method to Control the Introduced Class

The main objective is to present a method to convert the switched linear system with switching law constrained to logical state-input to PWA system. This method consists of state space meshing and allocation of each permissible input value to a region. To further explain the proposed concept, a specific state is considered in which the system possesses only a single logical input.

$$\dot{x} = A_i x + b_i, \qquad i = \varphi(x, s)\,, s = 0{,}1 \qquad (15)$$

The goal is to design logical variable in each region of state space:

$$s = v(x), \quad x \in R^n \qquad (16)$$

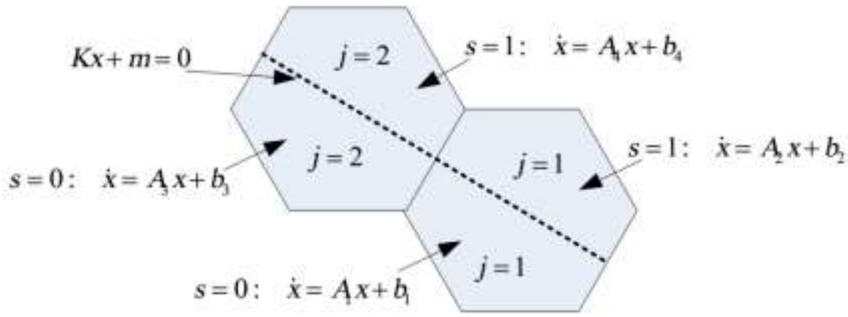

Figure 2. The investigated system after implementing the proposed method (in particular states: $M_s = 1 \; and \; M_{mesh} = 2$

Indeed, determining $v(x)$ function means ascertaining switching boundary in state plane. Considering a boundary in the form of a hyper-plane, logical variable is determined by:

$$s = step(Kx + m) \qquad (17)$$

Fig.2 illustrates the main idea of converting a switched linear system with switching law constrained to logical state-input to PWA system for a logical input. One side of the switching boundary is assumed to be logical input $s = 1$ and for the other side $s = 0$. Here, the design variable is coefficient matrixes of switching boundary (K, m). State variables are controlled in the system obtained from the proposed method by varying switching boundary. An advantage of this method is to alter the design realm from space of logical variables to continuous variables. The system produced by the proposed method is an unconventional PWA system, since the design variable appears in dynamics interfaces instead of being included in system dynamics. In the following, a computational method for stability analysis and controller design is presented for the selected system.

Stability of the Proposed Method

Firstly, a simple system including two subsystems without any constraints on the switching law is considered and will be further extended to a more general mode.

5-1. Stability of switched linear system without constraint on the switching law

Affine switched linear system is given by:

$$\dot{x} = A_i x + b_i, i = 1,2, \quad , x \in R^n, A_i \in R^{n\times n}, b_i \in R^n \qquad (18)$$

Here, i is the number of active subsystem and arbitrarily selected. Then, controller term of determining switching law boundary is written as follows:

$$i = step(Kx + m) + 1 \qquad (19)$$

System (18) accompanied by controller (19) are converted to a PWA system in the following form:

$$\begin{aligned} \dot{x} &= A_i + b_i, i = 1,2 \\ i = 1 &\Leftrightarrow Kx + m < 0 \\ i = 2 &\Leftrightarrow Kx + m > 0 \end{aligned} \qquad (20)$$

The generated system contains two meshes. This problem can be indirectly analyzed using the theorem in Ref (Rodrigues & How, 2003).and a series of BMI equations. A benefit of the proposed method is convex optimization and direct controller design by determining K and m.

Theorem 1:

System (18) under the controller (19) is globally asymptotically stable provided that LMI relations hold the following responses:

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & Y_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & Y_i \end{bmatrix} > 0, Y_i > 0 \qquad (21)$$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q_i - \alpha_i r\} & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & W_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & W_i \end{bmatrix} > 0 \qquad (22)$$

$$, W_i > 0$$

Here $i = 1,2$, $\alpha_i > 0$ is arbitrary constant scalar, $r \in R$, $q \in R^n$, $P \in R^{n\times n}$ are optimization variables, $0 < W_i = W_i^T \in R^{2\times 2}$, $0 < Y_i = Y_i^T \in R^{2\times 2}$ optimization variables with fully positive elements of inverse matrix and $\overline{g_i} \in R^2$, $\overline{H_i} \in R^{2\times n}$, and formed optimization variables are $\overline{H_1} = \begin{bmatrix} 0 \\ -K \end{bmatrix}$, $\overline{g_1} = \begin{bmatrix} 1 \\ m \end{bmatrix}$, $\overline{H_2} = \begin{bmatrix} 0 \\ K \end{bmatrix}$, and $\overline{g_2} = \begin{bmatrix} 1 \\ -m \end{bmatrix}$.

Proof: to prove the asymptotic stability, the following Lyapunov function is defined for all regions:

$$V(x) = x^T P x + 2q^T x + r \tag{23}$$

Where P is a symmetric matrix. Regarding the continuity of above Lyapunov function, satisfaction of its positivity and its derivative's negativity should be assessed. Lyapunov function (23) has a convergence rate of a_i in R_i if:

$$x \in R_i \Rightarrow \frac{d}{dt} V(x) < -\alpha_i V(x) \tag{24}$$

According to [1], $x\ \hat{I}\ R_i$ if:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \overline{H_i^T} Z_i \overline{H_i} & -\overline{H_i^T} Z_i \overline{g_i} \\ -\left(\overline{H_i^T} Z_i \overline{g_i}\right)^T & -\overline{g_i^T} Z_i \overline{g_i} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0 \tag{25}$$

Wherein Z_i is a symmetric matrix with positive elements and $\overline{H_1} = \begin{bmatrix} 0^{1\times n} \\ -K \end{bmatrix}$, $\overline{g_1} = \begin{bmatrix} 1 \\ m \end{bmatrix}$, $\overline{H_2} = \begin{bmatrix} 0^{1\times n} \\ K \end{bmatrix}$, and $\overline{g_2} = \begin{bmatrix} 1 \\ -m \end{bmatrix}$.

Associated with mesh description (25) and S-procedure [18, 19, 20], Lyapunov function is positive and its derivative is negative provided that matrixes $P > 0$, q, and r are existed as follows (Rodrigues & How, 2003):

$$\begin{bmatrix} P - \overline{H_i^T} Z_i \overline{H_i} & q + \overline{H_i^T} Z_i \overline{g_i} \\ q^T + \overline{g_i^T} Z_i \overline{H_i} & r - \overline{g_i^T} Z_i \overline{g_i} \end{bmatrix} > 0 \tag{26}$$

$$\begin{bmatrix} \left\{A_i^T P + PA_i + \alpha_i P + \overline{H_i^T} \Lambda_i \overline{H_i}\right\} & \left\{Pb_i + A_i^T q + \alpha_i q - \overline{H_i^T} \Lambda_i \overline{g_i}\right\} \\ \left\{-b_i^T P + q^T A_i + \alpha_i q^T - \overline{g_i^T} \Lambda_i \overline{H_i}\right\} & \left\{-2b_i^T q_i + \alpha_i r + \overline{g_i^T} \Lambda_i \overline{g_i}\right\} \end{bmatrix} < 0 \tag{27}$$

Z_i and Λ_i are matrixes with positive arrays in above relations. Furthermore, by satisfying these relations, stability is guaranteed and also it can be shown that the convergence rate in each R^i mesh is equal to α_i. Inequality (25) can be rewritten as:

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \overline{H_i} & -\overline{g_i} \\ \overline{H_i} & -\overline{g_i} \end{bmatrix} \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix} \begin{bmatrix} \overline{H_i} & -\overline{g_i} \\ \overline{H_i} & -\overline{g_i} \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0 \tag{28}$$

Associated with relation (28), relations (26) and (27) are rewritten as:

$$\begin{bmatrix} P & q \\ q^T & r \end{bmatrix} - \begin{bmatrix} \overline{H_i} & -\overline{g_i} \\ \overline{H_i} & -\overline{g_i} \end{bmatrix}^T \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix} \begin{bmatrix} \overline{H_i} & -\overline{g_i} \\ \overline{H_i} & -\overline{g_i} \end{bmatrix} > 0 \tag{29}$$

$$-\begin{bmatrix} A_i^T P + PA_i + \alpha_i P & Pb_i + A_i^T q + \alpha_i q \\ b_i^T P + q^T A_i + \alpha_i q^T & 2b_i^T q + \alpha_i r \end{bmatrix} - \begin{bmatrix} \overline{H}_i & -\overline{g}_i \\ \overline{H}_i & -\overline{g}_i \end{bmatrix}^T \begin{bmatrix} \Lambda_i & 0 \\ 0 & \Lambda_i \end{bmatrix} \begin{bmatrix} \overline{H}_i & -\overline{g}_i \\ \overline{H}_i & -\overline{g}_i \end{bmatrix} > 0 \quad (30)$$

In the following it will be assumed that Z_i and Λ_i not only do have positive elements but also are reversible. Their inverse matrices Z_i^{-1} and Λ_i^{-1} are positive definite. Based on schur complement(Boyd et all, 1994), it is concluded from above relations that:

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & Z_i^{-1} & 0 \\ \overline{H}_i^T & -\overline{g}_i & 0 & Z_i^{-1} \end{bmatrix} > 0, Z_i^{-1} > 0 \quad (31)$$

$> 0, Y_i > 0$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q - \alpha_i r\} & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & \Lambda_i^{-1} & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & \Lambda_i^{-1} \end{bmatrix} > 0 \quad (32)$$

$, \Lambda_i^{-1} > 0$

By defining $Z_i = Y_i^{-1}$ and $\Lambda_i = W_i^{-1}$, above relations are converted into LMI form as follows:

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & Y_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & Y_i \end{bmatrix} > 0, Y_i > 0 \quad (33)$$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q_i - \alpha_i r\} & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & W_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & W_i \end{bmatrix} > 0 \quad (34)$$

$, W_i > 0$

Therefore, stability is proved. This stability is asymptotic since $\alpha_i > 0$ and it is also global because Lyapunov function fulfils these features for R^n.

Point 1: If the theorem holds for different values of α_i such that $\overline{\alpha} = min\, \alpha_i \ (i = 1,2)$ is maximized, maximum convergence rate of the system will be provided.

Point 2: Positivity of elements of Y_i^{-1} and W_i^{-1} is equivalent to positivity of main diagonal elements and negativity of secondary diagonal elements of Y_i and W_i. The current conditions can be easily expressed in the form of LMI relations.

Piont 3: In this problem, adjustment parameters are $P, q, r, Y_i, W_i, \overline{H}_1, \overline{g}_1, \overline{H}_2, \overline{g}_2$ and it is aimed to calculate K and m.

Point 4: Theorem 1 is applicable to controller design for the PWA systems.

5-2. Theorem of stabilization and design for constrained switching between M subsystems

By assuming one logical input, the switched linear system with switching law constrained to logical state-input is expressed by:

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1,2,...,M, j = 1,2,...,M_{mesh} \\ &\varphi(.) = i, H_i^T x - g_i < 0, s = 0,1 \end{aligned} \tag{35}$$

Relation (35) with the assumption of using proposed controller is converted as follows:

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1,2,...,M, j = 1,2,...,M_{mesh} \\ &\varphi(.) = i, H_i^T x - g_i < 0, s = step(K_j x + m_j) \end{aligned} \tag{36}$$

In other words:

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1,2,...,M, j = 1,2,...,M_{mesh} \\ &\begin{cases} s = 1: \begin{bmatrix} H_j^T \\ -K_j \end{bmatrix} x - \begin{bmatrix} g_i \\ m_i \end{bmatrix} < 0 \\ s = 0: \begin{bmatrix} H_j^T \\ K_j \end{bmatrix} x - \begin{bmatrix} g_i \\ -m_j \end{bmatrix} < 0 \end{cases} \end{aligned} \tag{37}$$

When there is constraint on switching law and system has one logical input, theorem 1 is extended as follows:

Theorem 2: system

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1,2,...,M, j = 1,2,...,M_{mesh} \\ &\varphi(.) = i, H_i^T x - g_i < 0, s = 0,1 \end{aligned}$$

under the controller $s = step(K_j x + m_j)$ is globally asymptotically stable provided that LMI relations are fulfilled and have responses.

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & Y_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & Y_i \end{bmatrix} > 0, Y_i > 0 \tag{38}$$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q - \alpha_i r\} & -\overline{g}_i^T & -\overline{g}_i^T \\ \overline{H}_i & -\overline{g}_i & W_i & 0 \\ \overline{H}_i & -\overline{g}_i & 0 & W_i \end{bmatrix} > 0 \quad (39)$$

$, W_i > 0$

Here $\alpha_i > 0$ is arbitrary constant scalar, $r \in R$, $q \in R^n$, $P \in R^{n\times n}$ are optimization variables, $0 < W_i = W_i^T$, $0 < Y_i = Y_i^T$ optimization variables with fully positive elements of their inverse matrices, K and m are controller matrices, $\overline{g}_i \in R^3, \overline{H}_i \in R^{3\times n}$. Formed optimization variables are given by:

$$\begin{cases} i = 2j-1: \overline{H}_i = \begin{bmatrix} 0 \\ H_j^T \\ -K_j \end{bmatrix}, \overline{g}_i = \begin{bmatrix} 1 \\ g_j \\ m_j \end{bmatrix} \\ i = 2j: \overline{H}_i = \begin{bmatrix} 0 \\ H_j^T \\ K_j \end{bmatrix}, \overline{g}_i = \begin{bmatrix} 1 \\ g_j \\ -m_j \end{bmatrix} \end{cases}$$

Proof: It is adequately proved when for odd values of i, $\begin{bmatrix} g_i \\ m_j \end{bmatrix} \to g_i, \begin{bmatrix} H_j^T \\ -K_j \end{bmatrix} \to H_j^T$ and for even values of i, $\begin{bmatrix} g_i \\ -m_j \end{bmatrix} \to g_i, \begin{bmatrix} H_j^T \\ K_j \end{bmatrix} \to H_j^T$.

In the following, previous theorem is extended for an arbitrary number of logical inputs. Switched linear system with switching law constrained to logical state-input with M_s number of logical inputs is explained as:

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1,2,...,M \\ &\varphi(.) = i, H_i^T x - g_i < 0, j = 1,2,...,M_{mesh} \\ &S \in \{S_1, S_2, ..., S_{2^{M_s}}\} \end{aligned} \quad (40)$$

The controller is considered as:

$$S = msign(K_j x) \quad (41)$$

Where $msign$ function is the extension of step function and defined as follows:

$$msign: [U_{\min}, U_{\max}] \to \{0,1\}^{M_s}$$

$$msign(v)=\begin{cases}[0\ \ 0\ \ \dots\ \ 0]^T & U_{\min}\le v\le U_1\\ [0\ \ 0\ \ \dots\ \ 0]^T & U_1\le v\le U_2\\ . & .\\ . & .\\ . & .\\ [1\ \ 1\ \ \dots\ \ 1]^T & U_{2^{M_s}-1}\le v\le U_{\max}\end{cases} \quad (42)$$

$$U_{\min}<U_1<U_2<..<U_{2^{M_s}-1}<U_{\max}, U_0=U_{\min}, U_{2^{M_s}}=U_{\max}$$

Here, U_{min} and U_{max} are two arbitrary numbers and other indices of U_i are variable. With the assumption of controller (41), description of system (40), and definition of $msign$ function, it is concluded that:

$$\dot{x}=A_i x+b_i, i=1,2,...,M, j=1,2,....,M_{mesh}, l=1,2,...,2^{M_s}$$
$$\begin{bmatrix}H_j & -K_j^T & K_j^T\end{bmatrix}^T x-\begin{bmatrix}g_j^T & -U_{j,l-1}^T & U_{j,l}^T\end{bmatrix}^T<0 \quad (43)$$

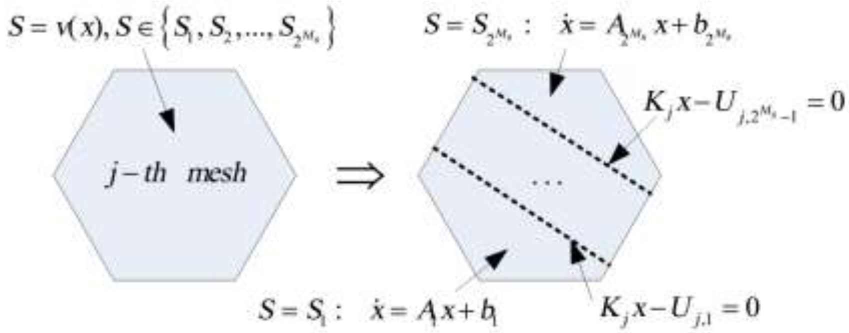

Figure 3. Allocation of logical input vector (S) in the proposed method (for $j-th$ mesh)

Fig.3 depicts the allocation of logical input vector (S) in the proposed approach for one of the investigated system meshes. All discussions of section 5-1 can be expressed by following matrices:

$$\overline{H}_i=\begin{bmatrix}0 & H_j & -K_j^T & K_j^T\end{bmatrix}^T,$$
$$\overline{U}_i=\begin{bmatrix}1 & g_j^T & -U_{j,l-1}^T & U_{j,l}^T\end{bmatrix}, i-1,2,...,M \quad (44)$$

And theorem 2 is extended as the following theorem.

Theorem 3: System (40) is globally asymptotically stable under controller (41) provided that the following LMI holds responses. Here, by solving optimization problem, controller matrices are then designed

$(K_j, U_{j1}, j = 1, 2, ..., M_{mesh})$.

$$U_{\min,j} < U_{j,1} < U_{j,2} < .. < U_{j,N-1} < +U_{\max}, \quad U_{j,0} = U_{\min}, U_{j,N} = U_{\max,j} \tag{45}$$

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{U}_i^T & -\overline{U}_i^T \\ \overline{H}_i & -\overline{U}_i & Y_i & 0 \\ \overline{H}_i & -\overline{U}_i & 0 & Y_i \end{bmatrix} > 0, Y_i > 0 \tag{46}$$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q - \alpha_i r\} & -\overline{U}_i^T & -\overline{U}_i^T \\ \overline{H}_i & -\overline{U}_i & W_i & 0 \\ \overline{H}_i & -\overline{U}_i^T & 0 & W_i \end{bmatrix} > 0, \quad W_i > 0 \tag{47}$$

Wherein $\alpha_i > 0$ has constant value, r, q, P are optimization variables, W_i, Y_i are optimization matrices that their inverse matrices have fully positive elements, and $\overline{U}_i, \overline{H}_i$ are formed optimization variables as follows:

$$\overline{H}_i = \begin{bmatrix} 0 & H & -K_j^T & K_j^T \end{bmatrix}^T,$$
$$\overline{U}_i = \begin{bmatrix} 1 & g_j^T & -U_{j,l-1}^T & U_{j,l}^T \end{bmatrix}^T,$$
$$i = 1, 2, ..., M, \; j = 1, 2, ..., M_{mesh}, \; l = 1, 2, ..., 2^{M_s}$$

Besides, $U_{\max,j}$ and $U_{\min,j}$ are assumed to be constant values.
Proof: System (40) with assuming controller (41) is expressed as:

$$\begin{aligned} &\dot{x} = A_i x + b_i, i = 1, 2, ..., M \\ &i = j : \overline{H}_i x - \overline{U}_i < 0 \\ &U_{\min} < U_1 < U_2 < ... < U_{M-1} < +U, \\ &U_0 = -U, U_N = U_{\min} \\ &\overline{H}_i = \begin{bmatrix} 0 \\ -\tilde{J}_j \\ -K_j \\ K_j \end{bmatrix}, \overline{U}_i = \begin{bmatrix} 1 \\ g_i \\ -U_{j,l-1} \\ U_{j,l} \end{bmatrix}, i = 1, 2, ..., M, \\ &j = 1, 2, ..., M_{mesh}, l = 1, 2, ..., 2^{M_s} \end{aligned} \tag{48}$$

Condition of describing $i-th$ mesh is given by [1]:

$$R_i = \left\{ x \left| \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} \overline{H}_i^T Z_i \overline{H}_i & -\overline{H}_i^T Z_i \overline{U}_i \\ -\left(\overline{H}_i^T Z_i \overline{U}_i\right)^T & \overline{U}_i^T Z_i \overline{U}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0 \right. \right\} \quad (49)$$

Where:

$$Z_i = \begin{bmatrix} Z_{00}^i & Z_{01}^i & . & . & Z_{0n}^i \\ Z_{10}^i & & & & . \\ . & & & & . \\ . & & & & . \\ Z_{1n}^i & & & & Z_{nn}^i \end{bmatrix}$$

And all elements of Z_i are positive. Furthermore, relation (41) can be rewritten as:

$$\begin{bmatrix} \overline{H}_i^T Z_i \overline{H}_i & -\overline{H}_i^T Z_i \overline{U}_i \\ -\left(\overline{H}_i^T Z_i \overline{U}_i\right)^T & \overline{U}_i^T Z_i \overline{U}_i \end{bmatrix} = \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix}^T \begin{bmatrix} 0.5Z_i & 0 \\ 0 & 0.5Z_i \end{bmatrix} \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix} > 0 \quad (50)$$

We consider Lyapunov function and its derivative as follows:

$$V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P & q \\ q & r \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad (51)$$

$$\frac{d}{dt} V(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i & Pb_i + A_i^T q \\ b_i^T P + q^T A_i & 2b_i^T q \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad (52)$$

Associated with relations (51, 52) and S-procedure, positivity condition of Lyanupov function with a convergence rate of α_i in R_i mesh is satisfied by presence of $P > 0, q, r$ matrices when all elements of Z_i and Λ_i are positive, such that:

$$\begin{bmatrix} P & q \\ q^T & r \end{bmatrix} - \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix}^T \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix} \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix} > 0 \quad (53)$$

Similarly, negativity condition for the derivative of Lyanupov function is satisfied as:

$$\begin{bmatrix} A_i^T P + PA_i + \alpha_i P & Pb_i + A_i^T q + \alpha_i q \\ b_i^T P + q^T A_i + \alpha_i q^T & 2b_i^T q + \alpha_i r \end{bmatrix} + \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix}^T \begin{bmatrix} Z_i & 0 \\ 0 & Z_i \end{bmatrix} \begin{bmatrix} \overline{H}_i & -\overline{U}_i \\ \overline{H}_i & -\overline{U}_i \end{bmatrix} > 0 > 0 \quad (54)$$

If Z_i and Λ_i are reversible and positive definite and based on schur complement relations (53) and (54) cane be rewritten as follows:

$$\begin{bmatrix} P & q & \overline{H}_i^T & \overline{H}_i^T \\ q^T & r & -\overline{U}_i^T & -\overline{U}_i^T \\ \overline{H}_i & -\overline{U}_i & Z_i^{-1} & 0 \\ \overline{H}_i & -\overline{U}_i & 0 & Z_i^{-1} \end{bmatrix} > 0, Y_i > 0 \quad (55)$$

$$\begin{bmatrix} \{-A_i^T P - PA_i - \alpha_i P\} & \{-Pb_i - A_i^T q - \alpha_i q\} & \overline{H}_i^T & \overline{H}_i^T \\ \{-b_i^T P - q^T A_i - \alpha_i q^T\} & \{-2b_i^T q - \alpha_i r\} & -\overline{U}_i^T & -\overline{U}_i^T \\ \overline{H}_i & -\overline{U}_i & \Lambda_i^{-1} & 0 \\ \overline{H}_i & -\overline{U}_i^T & 0 & \Lambda_i^{-1} \end{bmatrix} > 0, \quad (56)$$

$$W_i > 0$$

Bu defining $Z_i = Y_i^{-1}$ and $\Lambda_i = W_i^{-1}$, theorem is proved.

Point 5: A power electronics converter with more than one fully controllable semiconductor switch is an applicable example of theorem 3.

Point 6: In order to reach a response having steady-state error, it is needed to consider the switching equation that assumes $e = x_{ref} - x$, given by:

$$s = msign(K_j e) = msign(-K_j x + K_j x_{ref}) \quad (57)$$

In this condition, presented trend for theorem 3 is applicable by the following substitutions:

$$K_j x_{ref} \rightarrow m_j, -K_j \rightarrow K_j$$

Point 6: The following algorithm presents the process of stabilization and controller design for the studied class:

Step 1: The studied system in the form of switched linear system constrained to logical state-input is modelled based on relations (3) and (4), matrices A_i, b_i, C_i, d_i, and switching law $\varphi(i(t), s(t), x(t))$.

Step 2: Meshing process of space state and allocation of logical input to each mesh are carried out in this step (i.e. $s = msign(K_j x)$).

Step 3: Using extended form of theorem 3 and stabilizing controller design.

Simulation Results

Consider a Buk converter with the system equations and proposed stabilization controller:

$$\dot{x} = A_j x + b_i, i = 1,2,3,4$$
$$H_j x - g_j < 0, s = msign(K_j (x - x_{ref}))$$
$$H_1 = H_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}$$
$$H_3 = H_4 = \begin{bmatrix} +1 & 0 \end{bmatrix}$$
$$g_i = 0, j = 1,2$$

Table I defines matrices of former relation. Initial system includes two meshes extending to four meshes once the proposed controller is applied.

Therefore, variable $K_1 = K_2$ is designed by theorem 2 (for simplicity, only one switching boundary is employed for both meshes).

$$j = 1 \Leftrightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} x > 0$$
$$j = 2 \Leftrightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} x > 0$$

And

$$\overline{H}_1 = \overline{H}_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ & -K_1 \end{bmatrix}, \overline{H}_2 = \overline{H}_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ & K_1 \end{bmatrix},$$
$$m_1 = m_3 = \begin{bmatrix} 1 \\ 0 \\ -K_1 x_{ref} \end{bmatrix}, m_2 = m_4 = \begin{bmatrix} 0 \\ 1 \\ K_1 x_{ref} \end{bmatrix}$$

Simulation has been conducted using parameters presented in Table 2. By solving relevant LMI relations through Robust Control toolbox in MATLAB software, the following result is obtained:

$$K_1 = K_2 = 10^{-8} \times \begin{bmatrix} -0.4263 & -0.2185 \end{bmatrix}$$

Obtained simulation results are illustrated in Figs. 4 and 5. Fig. 4 displays outlet variable changes. Response possesses the steady-state error and a significant transpose. However, the proposed approach in this paper is aimed to investigate stabilization while there is no intention to examine the transient response optimization problem. Fig. 5 depicts the state plane accompanied by designed and inherent boundary of the system. Logical input is I and 0 at the top and bottom of designed boundary, respectively. As it can be observed in this figure, state space is divided into 4 meshes where each of them is one of the four active subsystems.

Table 1. Descriptive matrices for Buk inverter along with activation condition of each subsystem

i	1	2	3	4
A_i	$\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix}$	$\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{bmatrix}$
b_i	$\begin{bmatrix} \frac{V_g}{L} & 0 \end{bmatrix}^T$	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0 \quad 0]^T$
$\varphi(.)$	$s=1$ $[0 \quad 1]x>0$	$s=0$ $[1 \quad 0]x>0$	$s=1$ $[1 \quad 0]x<0$	$s=0$ $[1 \quad 0]x<0$
C_i	$[0 \quad 1]$	$[0 \quad 1]$	$[0 \quad 1]$	$[0 \quad 1]$

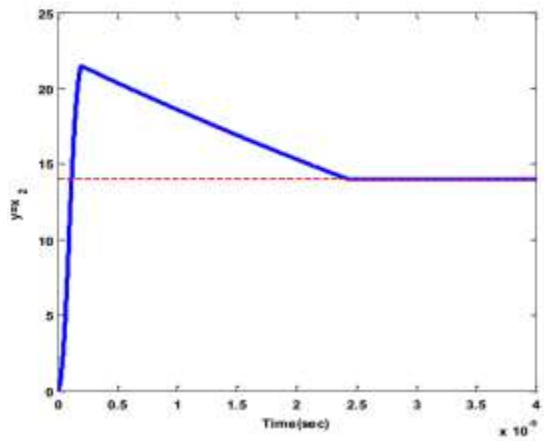

Figure 4. Outlet variations diagram

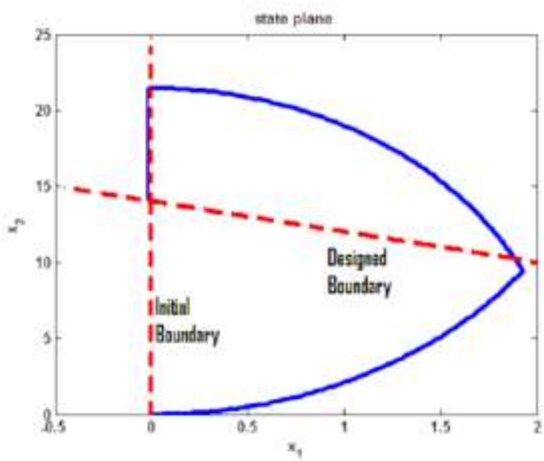

Figure 5. Diagram of state plane accompanied by inherent and designed boundary

Table 2. Descriptive matrices for Buk inverter along with activation condition of each subsystem

Variable	Symbol	Value	Unit
Load resistance	R	1000	Ω
inductor filter	L	10μ	H
Outlet capacitor	C	$100n$	F
Supply voltage	V_g	24	V
Desired outlet voltage	$x_{2,ref}$	14	V
Desired inductor current	$x_{1,ref}$	0.014	A

Conclusion

A new method was introduced in this paper for stabilization and controller design of switched linear systems with constrained switching law. In order to determine logical input, regions of state plane are meshed and a logical input is allocated to each of the meshed region. Implementing the proposed method is simple and its design trend is based on a set of LMI relations. Furthermore, theorems for stabilization and controller design for the studied class are investigated and proved. It is suggested for further research to improve transient state by adding appropriate LMI constraints.

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