Artículo de investigación

Quadrature Formulas for the Calculation of the Riemann-Liuville Fractional Integral

Fórmulas de cuadratura para el cálculo de la integral fraccional de Riemann-Liouville

Fórmulas de Quadratura para o Cálculo do Integral Fracionário de Riemann-Liouville

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Abstract

The quadrature formulas for the fractional Riemann-Liouville integral are investigated in this article. A linear operator is introduced that associates $\forall \varphi(x) \in C[a,b]$ a polynomial $P_n(\varphi; x) \in H_n$ satisfying the condition $(I^{\alpha}_{a+}P_n)(x_j) = (I^{\alpha}_{a+}\varphi)(x_j)$, $j = \overline{0,n}$, where x_j – are Chebyshev points. The integrand is approximated by an algebraic polynomial. The formula of the remainder term for the quadrature formula is derived. The quadrature formulas obtained are verified using the Wolfram Mathematica computer algebra system.

Keywords: quadrature formula, fractional integration, fractional calculation, fractional Riemann-Liouville integral.

Resumen

Las fórmulas en cuadratura para la integral fraccional de Riemann-Liouville se investigan en este artículo. Se introduce un operador lineal que asocia $\forall \varphi(x) \in C[a,b]$ un polinomio $P_n(\varphi; x) \in H_n$ que satisface la condición $(I^{\alpha}_{a+}P_n)(x_j) = (I^{\alpha}_{a+}\varphi)(x_j)$, $j = \overline{0,n}$ donde: x_j son puntos de Chebyshev. El integrando es aproximado por un polinomio algebraico. Se deriva la fórmula del término restante para la fórmula de cuadratura. Las fórmulas en cuadratura obtenidas se verifican usando el sistema de álgebra computacional Wolfram Mathematica.

Palabras claves: fórmula de cuadratura, integración fraccional, cálculo fraccional, integral fraccional de Riemann-Liouville

Resumo

As fórmulas de quadratura para a integral fracionária de Riemann-Liouville são investigadas neste artigo. Um operador linear é introduzido que associa $\forall \varphi(x) \in C[a,b]$ um polinômio $P_n(\varphi; x) \in H_n$ satisfazendo a condição $(I^{\alpha}_{a+}P_n)(x_j) = (I^{\alpha}_{a+}\varphi)(x_j)$, $j = \overline{0,n}$ onde x_j - são os pontos de Chebyshev. O integrando é aproximado por um polinômio algébrico. A fórmula do termo restante para a fórmula de quadratura é derivada. As fórmulas de quadratura obtidas são verificadas usando o sistema de álgebra computacional da Wolfram Mathematica.

Palavras-chave: fórmula de quadratura, integração fracionária, cálculo fracional, integral fracionária de Riemann-Liouville.

Introduction

Fractional integration and differentiation is the extension of the operations of integration and differentiation to the case of fractional orders (Ortigueira & Machado, 2015). In recent years, fractional integro-differentiation has been used to solve many applied problems in mechanics, control theory, biology, etc. A large number of studies are devoted to solving, with the help of fractional integrals, those problems in various disciplines that can not be solved by conventional means. For example, in (Gonzalez & Petras, 2015; Liu et al, 2015), the use of fractional calculus in signal processing is considered (DEMIYEVA, 2017; Villalobos Antúnez, 2015).

In (Evans et al, 2017), the use of fractional integrals and derivatives in medicine is studied. The applications of fractional calculus in mechanics are given in (Bachher et al, 2015; Ezzat et al, 2015). The solution of fractional differential equations is considered in (Kaplan & Bekir, (2016; Mirzaee & Alipour, 2018; Zeng et al, 2017; Pospisil & Skripkova, 2016; Chadha & Pandey, 2017).

We consider the necessary definitions.
For an n- multiple integral, the following formula is:

$$\int_a^x dx \int_a^x dx \ldots \int_a^x \varphi(x)dx = \frac{1}{(n-1)!}\int_a^x (x-1)^{n-1}dx$$

.

Where $(n-1)! = \Gamma(n)$
The right-hand part of the formula makes sense also for noninteger values of n.

Definition 1. Let $\varphi(x) \in C[a,b]$, the $[a,b]$ continuous function. Integrals

$$(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, x > a$$

,and

$$(I_{b-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, x < b$$

,

where $\alpha > 0$, are called Riemann-Liouville α fractional integrals. The first of them is called left-handed, and the second is called the right-handed integral. Operators I_{a+}^{α}, I_{b-}^{α} are called fractional integration operators.

We will consider left-handed fractional integrals with order $0 < \alpha < 1$.
The operators of fractional integration have the attribute

$$I_{a+}^{\alpha} I_{a+}^{\beta}\varphi = I_{a+}^{\alpha+\beta}\varphi, \qquad I_{b-}^{\alpha} I_{b-}^{\beta}\varphi = I_{b-}^{\alpha+\beta}\varphi,$$

$\alpha > 0, \beta > 0$

Definition 2. For $f(x)$ function, set on $[a,b]$ interval, every equation

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt$$

,

$$(D_{b-}^{\alpha} f)(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^{\alpha}}dt$$

, is called the fractional derivative of order, α, $0 < \alpha < 1$. The first of them is called left-handed, and the second is called the right-handed derivative. These fractional derivatives are also called Riemann-Liouville derivatives.

Note that, $D_{a+}^{\alpha} I_{a+}^{\alpha}\varphi \equiv \varphi$, but $I_{a+}^{\alpha} D_{a+}^{\alpha}\varphi$ does not always coincide with $\varphi(x)$.

Fractional integrals and Riemann-Liouville derivatives are investigated by specialists in many works, for example, in (Ezz-Eldien et al, 2018) they are studied in connection with the solution of problems of fractional calculus of variations.
The expansion of the fractional Riemann-Liouville derivative is given in (Baleanu et al, 2017). Fractional integral inequalities are considered in (Sudsutad et al, 2016; Anastassiou, 2017) and the continuity and boundedness of the fractional Riemann-Liouville integral operator are studied.

Definition 3. The Pochhammer symbol $(z)_n$ for n integers is defined by $(z)_n = z(z+1)\ldots(z+n-1)$, $n = 1,2,\ldots$, $(z)_0 \equiv 1$.

where $(z)_n = (-1)^n(1-n-z)_n$, $(1)_n = n!$

Definition 4. Gamma function $\Gamma(z)$ is the Euler integral of the second kind

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$$\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx$$, $\operatorname{Re} z>0$, where $x^{z-1}=e^{(z-1)\ln x}$.

The reduction formula $\Gamma(z+1)=z\Gamma(z)$, $\operatorname{Re} z>0$ is also use attribute $\Gamma(n)=(n-1)!$, $n=1,2,\ldots$, $\Gamma(1)=1$

Definition 5. Beta function is the $\mathrm{B}(z,\varpi)=\int_0^1 x^{z-1}(1-x)^{\varpi-1}dx$ integral, $\operatorname{Re} z>0$, $\operatorname{Re}\varpi>0$ (the Euler integral of the first kind)

Beta function is expressed by the gamma function by the formula $\mathrm{B}(z,\varpi)=\dfrac{\Gamma(z)\Gamma(\varpi)}{\Gamma(z+\varpi)}$

Derivation of Quadrature Formula for the Riemann-Liouville Fractional Integral Calculation

The methods for approximation of fractional integrals, in particular Riemann-Liouville integrals, are widely studied issue at the present time. For example, in (Colinas-Armijo & Di Paola, 2018) we consider the step-by-step approximation of a fractional integral. In (Toufik & Atangana, 2017) approximate formulas for the solution of linear and nonlinear fractional differential equations are given, in (Zedan et al, 2017) - an approximation of a system of fractional integro-differential equations. Earlier in (Galimyanov et al, 2017) we gave quadrature formulas for the fractional Weyl integral.

Definition 6. Let us consider a definite integral $\int_a^b p(x)f(x)dx$, where $a,b\in R=(-\infty,\infty)$, $f(x)$ is a certain continuous function, $p(x)$ is a weight function, i.e. $p(x)\geq 0$ throughout (a,b) and is equal 0 only on a set of a zero measure and $\exists\int_a^b p(x)dx<+\infty$.

Then, if this integral exists, but is not exactly calculated, then we can use formula

$$\int_a^b p(x)f(x)dx\approx\sum_{k=1}^n A_k f(x_k)$$

, where x_k are some points of (a,b). The formula is called the quadrature formula.

Let construct a quadrature formula for calculating the fractional Riemann-Liouville integral.

Let $x_j=\cos\dfrac{2j+1}{2n+2}\pi$, $j=\overline{0,n}$ – are Chebyshev points of the first kind. (1)

We note by a P_n linear operator that associates $\forall\varphi(x)\in C[a,b]$ a polynomial $P_n(\varphi;x)\in H_n$ satisfying the condition, $(I_{a+}^\alpha P_n)(x_j)=(I_{a+}^\alpha\varphi)(x_j)$, $j=\overline{0,n}$. (2).

We approximate the integrand $\varphi(x)$ by an algebraic polynomial of the form

$$P_n(x)=\sum_{k=0}^n a_k(x-a)^k \quad (3)$$

Approximation by a polynomial is often used in solving approximation problems, for example, in (Nemati & Yousefi, 2017) we can see the application of such a method.

We calculate $(I_{a+}^\alpha P_n)(x)$. We substitute the representation of a polynomial in a fractional integral and transform it.

$$(I_{a+}^\alpha P_n)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{P_n(t)}{(x-t)^{1-\alpha}}dt=\frac{1}{\Gamma(\alpha)}\int_a^x\sum_{k=0}^n a_k(t-a)^k\frac{dt}{(x-t)^{1-\alpha}}=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=0}^n a_k\int_a^x(t-a)^k(x-t)^{\alpha-1}dt.$$

We make the following change of variables

$$t=x-(x-a)\delta,\quad \delta=\frac{x-t}{x-a},$$

$$dt=-(x-a)d\delta$$

and transform the resulting calculation

$$(I_{a+}^\alpha P_n)(x)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^n a_k\int_1^0(x-(x-a)\delta-a)^k(x-x+(x-a)\delta)^{\alpha-1}(-(x-a))d\delta=$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^n a_k\int_0^1(x-a)^k(1-\delta)^k(x-a)^{\alpha-1}\delta^{\alpha-1}(x-a)d\delta=$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^n a_k(x-a)^{k+\alpha}\int_0^1(1-\delta)^k\delta^{\alpha-1}d\delta.$$

In the integrand to the k degree we add and subtract 1.

$$(I_{a+}^{\alpha}P_n)(x)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\int_0^1(1-\delta)^{(k+1)-1}\delta^{\alpha-1}d\delta$$

Note that we have obtained a B-function. Then, taking into account the properties of the B-function, we have the following

$$(I_{a+}^{\alpha}P_n)(x)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(k+1+\alpha)}=$$
$$\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}.$$

Thus,

$$(I_{a+}^{\alpha}P_n)(x)=\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

.

We find the unknowns $\{a_k\}_0^n$ from condition (2), where the x_j points are described in formula (1).

We obtain the following system of linear algebraic equations with respect to $\{a_k\}_0^n$

$$\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}=\frac{1}{\Gamma(\alpha)}\int_a^{x_j}\frac{\varphi(t)}{(x_j-t)^{1-\alpha}}dt,$$

$j=\overline{0,n}$. (4)

It follows from the general theory of linear algebraic systems that this system will have a unique solution if its determinant is different from zero. Therefore, we consider the determinant of this system:

$$\Delta=\begin{vmatrix}(x_0-a)^{\alpha}\frac{\Gamma(1)}{\Gamma(1+\alpha)} & (x_0-a)^{\alpha+1}\frac{\Gamma(1)}{\Gamma(2+\alpha)} & \dots & (x_0-a)^{\alpha+n}\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\\ \dots & \dots & \dots & \dots\\ (x_n-a)^{\alpha}\frac{\Gamma(1)}{\Gamma(1+\alpha)} & (x_n-a)^{\alpha+1}\frac{\Gamma(1)}{\Gamma(2+\alpha)} & \dots & (x_n-a)^{\alpha+n}\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\end{vmatrix}$$

.

We use the $\Gamma(z)$ attributes of a function: $\Gamma(\alpha+n)=\Gamma(\alpha)(\alpha)_n$, $\Gamma(n)=(n-1)!$, $\Gamma(1)=1$.

$$\Delta=\begin{vmatrix}(x_0-a)^{\alpha}\frac{0!}{\Gamma(\alpha)(\alpha)_1} & (x_0-a)^{\alpha+1}\frac{1!}{\Gamma(\alpha)(\alpha)_2} & \dots & (x_0-a)^{\alpha+n}\frac{n!}{\Gamma(\alpha)(\alpha)_{n+1}}\\ \dots & \dots & \dots & \dots\\ (x_n-a)^{\alpha}\frac{0!}{\Gamma(\alpha)(\alpha)_1} & (x_n-a)^{\alpha+1}\frac{1!}{\Gamma(\alpha)(\alpha)_2} & \dots & (x_n-a)^{\alpha+n}\frac{n!}{\Gamma(\alpha)(\alpha)_{n+1}}\end{vmatrix}$$

It can be seen that the determinant contains a common factor $\frac{1}{\alpha\Gamma(\alpha)}$ that can be taken out, also each line contains a common factor $(x_0-a)^{\alpha}$, $(x_1-a)^{\alpha}$, …, $(x_n-a)^{\alpha}$, respectively, and each column contains a common factor $\frac{1!}{\alpha+1}$,…, $\frac{n!}{(\alpha+1)\dots(\alpha+n)}$, respectively. Therefore, we obtain

$$\Delta=(x_0-a)^{\alpha}\dots(x_n-a)^{\alpha}\left(\frac{1}{\alpha\Gamma(\alpha)}\right)^n\frac{1!}{\alpha+1}\dots\frac{n!}{(\alpha+1)\dots(\alpha+n)}\begin{vmatrix}1 & (x_0-a) & \dots & (x_0-a)^n\\ \dots & \dots & \dots & \dots\\ 1 & (x_n-a) & \dots & (x_n-a)^n\end{vmatrix}$$

.

The remaining determinant $\begin{vmatrix}1 & (x_0-a) & \dots & (x_0-a)^n\\ \dots & \dots & \dots & \dots\\ 1 & (x_n-a) & \dots & (x_n-a)^n\end{vmatrix}$ is Vandermonde's determinant.

Definition 7. Vandermonde's determinant is

$$\Delta=\begin{vmatrix}1 & x_0 & \dots & x_0^{\,n}\\ \dots & \dots & \dots & \dots\\ 1 & x_n & \dots & x_n^{\,n}\end{vmatrix}=\prod_{n\ge k\ge m\ge 0}(x_k-x_m)\neq 0$$

.

It follows that the system of linear algebraic equations (4) has a unique solution - unknowns $\{a_k\}_0^n$ are uniquely determined from this system.

As a result, we obtain the following quadrature formula

$$\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt\approx\sum_{k=0}^{n}a_k(x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

.

Definition 8.

The variable $R_n(f)=\int_a^b p(x)f(x)dx-\sum_{k=1}^{n}A_kf(x_k)$ at $C(a,b)$ is called the remainder (the remainder term) of the quadrature formul a $\int_a^b p(x)f(x)dx\approx\sum_{k=1}^{n}A_kf(x_k)$.

For our case, the remainder of the quadrature formula will be $R_n(f) = I_{a+}^{\alpha}(\varphi, x) - I_{a+}^{\alpha}(P_n, x)$.

$(I_{a+}^{\alpha} P_n)(x)$ is an interpolation polynomial, so we use the representation of the remainder term for interpolation polynomials.

Statement. Let a $f(x) \in C[a,b]$ function have continuous or at least bounded derivatives of n-kind, that is $\exists f^{(n)}(x) \in [a,b]$, then the remainder term can be represented in the form $r_n(x) = \frac{f^{(n)}(\xi)}{n!}\omega_n(x)$, where $\xi \in (a,b)$, where is a certain $\omega_n(x) = (x-x_0)(x-x_1)...(x-x_n)$.

Therefore, we get $R_n(\varphi) = \frac{((I_{a+}^{\alpha}\varphi)(x))^{(n)}(\xi)}{n!}\omega_n(x)$ if the function has continuous derivatives of n-kind, that is $\exists \varphi^{(n)}(x) \in [a,b]$

We will show this. We have

$$\frac{d^n}{dx^n}(I_{a+}^{\alpha}\varphi)(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt\right)$$

We integrate by parts.

$$\frac{d^n}{dx^n}(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\frac{d^n}{dx^n}\left(-\varphi(x)(x-t)^{\alpha}\Big|_a^x - \int_a^x (x-t)^{\alpha}\varphi'(t)dt\right) =$$

$$= \frac{1}{\Gamma(\alpha)}\frac{d^n}{dx^n}\left(\varphi(a)(x-a)^{\alpha} - \int_a^x (x-t)^{\alpha}\varphi'(t)dt\right)$$

.

Separate one derivative and differentiate the expression in brackets.

$$\frac{d^n}{dx^n}(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\frac{d^{n-1}}{dx^{n-1}}\left(\varphi(a)\frac{\alpha-1}{(x-a)^{1-\alpha}} - (\alpha-1)\int_a^x \frac{\varphi'(t)}{(x-a)^{1-\alpha}}dt\right)$$

Again we integrate by parts and then, separating one more derivative, we differentiate the resulting expression, and so we continue until we exhaust all n derivatives.

As a result, we get

$$\frac{d^n}{dx^n}(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\left(\sum_{k=1}^{n}\varphi^{(k)}(a)\frac{(\alpha-1)^k(\alpha-2)...(\alpha-k)}{(x-a)^{k-\alpha}} - (\alpha-1)^n\int_a^x \frac{\varphi^{(n)}(t)}{(x-a)^{1-\alpha}}dt\right)$$

Theorem. To calculate the fractional Riemann-Liouville integral, we can use the quadrature formula

$$\frac{1}{\Gamma(\alpha)}\int_a^{x_j}\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt \approx \sum_{k=0}^{n} a_k (x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

,(5)

where $\{a_k\}_0^n$ are determined from the system of linear algebraic equations (4).

For a $\varphi(x) \in C^{(n)}[a,b]$ function, the formula has an error

$$R_n(\varphi) = \frac{((I_{a+}^{\alpha}\varphi)(x))^{(n)}(\xi)}{n!}\omega_n(x)$$

.

Calculations in the Worlfram Mathematica System

The quadrature formulas obtained were verified using the Wolfram Mathematica computer algebra system. The calculations were performed for various functions for different values of α and n. Here are some results:

For the calculations we took the function $y = x^2$, $\alpha = 0{,}2$, $n = 5$. Table I contains a part of the results obtained by the quadrature formula and the formula given in Definition 1 for x values from the interval $(-1;1]$ with a step of 0.1.

Table I

x	Values of I_{a+}^{α} calculated by the quadrature formula	Values of I_{a+}^{α} calculated by the main formula
-0,9	0,57700216320663	0,57786520325503
-0,8	0,54988446470939	0,55017061596212
…	…	…
0,1	0,09237347341898	0,09250704050211
…	…	…
0,9	0,70297750691299	0,70358334197922
1	0,87136940841457	0,87196166393050

At the half-interval $(-1;1]$, the maximum magnitude of the difference between the values calculated from the quadrature formula (5) and the

$$(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt$$

formula is 0, 00086304. For $\alpha = 0{,}5$, $n = 5$, this value is 2.72537 · 10-8, for $\alpha = 0{,}5$,

$n=10$ – 7,94875·10-9, and for $\alpha=0{,}9$, $n=5$ this value is – 3,35287·10-14.

For the function $y=\sin x$ where $\alpha=0{,}5$, $n=5$, at the half-interval $(-1;1]$, the maximum magnitude of the difference between the values calculated from the quadrature formula (5) and the $(I_{a+}^{\alpha}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt$ formula is 1.5436 · 10-6.

Summary

The values obtained with the derived quadrature formulas are verified using the Wolfram Mathematica computer algebra system. It is noted that the error in calculating the quadrature formula decreases with increasing of α, $0<\alpha<1$.

Conclusions

Thus, quadrature formulas for the Riemann-Liouville fractional integral are investigated. A linear operator is introduced that assigns a polynomial to any function. The approximation of the integrand by an algebraic polynomial is considered. The remainder term formula for the quadrature formula is derived.

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Bibliography

Anastassiou, G. (2017). On left multidimensional Riemann-Liouville fractional integral // Journal of Computational Analysis and Applications, 23(2), 239-248.

Bachher, M., Sarkar, N., Lahiri, A. (2015). Fractional order thermoelastic interactions in an infinite porous material due to distributed time-dependent heat sources // Meccanica, 50 (8), 2167-2178

Baleanu D., Agarwal P., Parmar R. K., Alqurashi M. M., Salahshour S. (2017). Extension of the fractional derivative operator of the Riemann-Liouville // Journal of Nonlinear Sciences and Applications, 10 (6) 2914-2924.

Chadha, A., Pandey, D. N. (2017). Approximations of Solutions of a Neutral Fractional Integro-Differential Equation // Differential Equations and Dynamical Systems, 25 (1), 117-133.

Colinas-Armijo N., Di Paola M. (2018). Step-by-step integration for fractional operators // Communications in Nonlinear Science and Numerical Simulation. , 59, 292-305.

DEMIYEVA A.G. (2017). Custom in the system of sources of entrepreneurial law of the Russian Federation and international practice (by the example of the Eeu), Astra Salvensis, Supplement No. 10, p. 43-49.

Evans R. M., Katugampola U. N., Edwards D. A. (2017). Applications of fractional calculus in solving Abel-type integral equations: Surface-volume reaction problem // Computers & Mathematics with Applications. 73 (6), 1346-1362.

Ezzat, M. A., Sabbah, A. S., El-Bary, A. A., Ezzat, S. M. (2015). Thermoelectric Viscoelastic Fluid with Fractional Integral and Derivative Heat Transfer // Advances in Applied Mathematics and Mechanics, 7 (4), 528-548

Ezz-Eldien S. S., Doha E. H., Bhrawy A. H., El-Kalaawy A. A., Machado J. A. T. (2018). A new operational approach for solving fractional variational problems depending on indefinite integrals // Communications in Nonlinear Science and Numerical Simulation. 57, 246-263.

Galimyanov A. F. Gilemzyanov A.F., Minnegalieva C.B. (2017). Square formulas for Weil fractional integral based on trigonometric polynom. // J. Fundam. Appl. Sci., 9(1s), 1933-1944.

Gonzalez E. A., Petras I. (2015). Advances in Fractional Calculus: Control and Signal Processing Applications // 2015 16th International Carpathian Control Conference (Iccc). 147-152.

Kaplan, M., Bekir, A. (2016). A novel analytical method for time-fractional differential equations // Optik, 127 (20), 8209-8214

Liu, D. Y., Gibaru, O., Perruquetti, W., Laleg-Kirati, T. M. (2015). Fractional Order Differentiation by Integration and Error Analysis in Noisy Environment // Ieee Transactions on Automatic Control, 60(11) 2945-2960

Mirzaee, F., Alipour, S. (2018). Approximate solution of nonlinear quadratic integral equations of fractional order via piecewise linear functions // Journal of Computational and Applied Mathematics, 331, 217-227

Nemati, A., Yousefi, S. A. (2017). A numerical scheme for solving two-dimensional fractional optimal control problems by the Ritz method combined with fractional operational matrix // Ima Journal of Mathematical Control and Information, 34 (4), 1079-1097

Ortigueira, M. D., Machado, J. A. T. (2015). What is a fractional derivative? // Journal of Computational Physics, 293, 4-13

Pospisil, M, Skripkova, L. P. (2016). Sturm's theorems for conformable fractional differential equations // Mathematical Communications, 21 (2), 273-281

Sudsutad, W., Ntouyas, S. K., Tariboon, J. (2016). On mixed type Riemann-Liouville and Hadamard fractional integral inequalities // Journal of Computational Analysis and Applications, (21), 299-314.

Toufik, M., Atangana, A. (2017). New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models // European Physical Journal Plus, 132 (10)

Villalobos Antúnez J.V. (2015). Tecnociencia, derecho y sociedad. Pilares de una modernidad inacabada, Opción, vol. 31, núm. 76, enero-abril, pp. 5-12.

Zedan, H. A., Tantawy, S. S., Sayed, Y. M. (2017). New Solutions for System of Fractional Integro-Differential Equations and Abel's Integral Equations by Chebyshev Spectral Method // Mathematical Problems in Engineering, 13

Zeng, F. H. , Zhang, Z. Q., Karniadakis, G. E. (2017). Second-order numerical methods for multi-term fractional differential equations: Smooth and non-smooth solutions // Computer Methods in Applied Mechanics and Engineering, 327, 478-502