Artículo de investigación

Quadrature formula for singular integral computation of special type

Fórmula de cuadratura para el cálculo integral singular de tipo especial
Fórmula quadratura para computação integral singular de tipo especial

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Abstract

In this paper we develop the quadrature formula for the singular integral with the Cauchy kernel from the fractional Riemann-Liouville integral. The derivation is based on the quadrature formula obtained previously to calculate the Riemann-Liouville fractional integral. During the development of the quadrature formula to calculate a singular integral with the Cauchy kernel, we use the formulas for the exact calculation of Cauchy type integral principal value. The formula for the remainder of the quadrature formula is derived. The error estimation was performed. A special case of the developed quadrature formula was considered. The calculations were performed in Wolfram Mathematica system.

Keywords: fractional integral, Riemann-Liouville integral, fractional integration, fractional differentiation, quadrature formula, fractional calculus, singular integral.

Resumen

En este artículo desarrollamos la fórmula para la integral singular con el núcleo de Cauchy a partir de la integral fraccional de Riemann-Liouville. La derivación se basa en la fórmula de cuadratura obtenida anteriormente para calcular la integral fraccional de Riemann-Liouville. Durante el desarrollo de la fórmula de cuadratura para calcular una integral singular con el núcleo de Cauchy, usamos las fórmulas para el cálculo exacto del valor principal integral del tipo de Cauchy. Se deriva la fórmula para el resto de la fórmula de cuadratura. La estimación del error se realizó. Se consideró un caso especial de fórmula de cuadratura desarrollada. Los cálculos se realizaron en el sistema Wolfram Mathematica.

Palabras claves: Integral fraccional, integral de Riemann-Liouville, integración fraccional, diferenciación fraccional, fórmula en cuadratura, cálculo fraccional, integral singular.

Resumo

Fórmula para a integral singular com o núcleo de Cauchy da integral fracionária de Riemann-Liouville. A derivação é baseada na fórmula de quadratura obtida anteriormente para calcular a integral fracional de Riemann-Liouville. Durante o desenvolvimento da fórmula de quadratura para calcular uma integral singular com o kernel de Cauchy, usamos as fórmulas para o cálculo exato do valor principal integral do tipo Cauchy. A fórmula para o restante da fórmula de quadratura é derivada. A estimativa de erro foi realizada. Um caso especial da fórmula de quadratura desenvolvida foi considerado. Os cálculos foram realizados no sistema Wolfram Mathematica.

Palavras-chave: Integral fracionário, integral de Riemann-Liouville, integração fracional, diferenciação fracional, fórmula de quadratura, cálculo fracionário, integral singular.

Introduction

Fractional derivatives and the integrals of real or complex order are studied and applied in fractional calculation. It has a long history, many well-known mathematicians influenced the development of fractional integral differentiation. The fractional calculus is being developed intensively at the present time (Parovik, 2017; Villalobos Antúnez, & Popper, 2017). Fractional integrals and derivatives have a peculiar specificity (Labora & Rodriguez-Lopez, 2017).

If $\varphi(x) \in C[a,b]$, the integrals

$$(I_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, x > a,$$

and

$$(I_{b-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, x < b,$$

where $\alpha > 0$, are called the integrals of fractional order α according to Riemann-Liouville. Earlier, we developed the quadrature formulas of the interpolation type to calculate the fractional Riemann-Liouville integral $(I_{a+}^{\alpha}\varphi)(x)$ and for the fractional Weyl integral (Galimyanov et al, 2017; Golovchun et al, 2017).

Numerous theoretical and applied problems are reduced to integral and integral-differential equations with various regular and singular integrals (Micula, 2018; Gorskaya & Galimyanov, 2014; Pshu, 2017). Since the integrals are calculated precisely only in rare cases, the development of approximate methods for their computation with an appropriate theoretical justification is of great importance for applications (Gorskaya & Galimyanov, 2017; Zhu et al, 2018). The singular integral with the Cauchy kernel has the following form:

$$(S\varphi)(x) = \frac{1}{\pi}\int_a^b \frac{\varphi(t)}{t-x}dt$$

. Next, we will consider the development of the quadrature formula for the integral, using fractional integral values.

Methods

One way to obtain quadrature formulas for integrals is to approximate their density by various analytical devices. In this paper we derive the quadrature formula to calculate the singular integral with the Cauchy kernel from the fractional Riemann-Liouville integral. To do this, the fractional Riemann-Liouville integral is computed by the quadrature formula derived earlier. To verify the resulting formula, the calculations are carried out in the computer algebra system Wolfram Mathematica.

Results and Discussion

Fractional integrals and derivatives, Riemann-Liouville fractional integrals are used to solve various problems of fractional integral-differentiation (Agarwal et al, 2017; Li & Xiao, 2017; Grinko & Kilbas, 2007).

Let's consider $S(I_{-1+}^{\alpha}(\varphi))(x)$, where

$$(S\varphi)(x) = \frac{1}{\pi}\int_a^b \frac{\varphi(t)}{t-x}dt$$

is the singular integral with the Cauchy kernel and develop the quadrature formula for this integral, based on the quadrature formula obtained earlier for the Riemann-Liouville fractional integral

$$\frac{1}{\Gamma(\alpha)}\int_a^{x_j} \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt \approx \sum_{k=0}^{n} a_k (x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

Let's take a particular case, when $a=-1$, $b=1$.

Then we obtain the following:

$$(I_{-1+}^{\alpha}\varphi_n)(x) = \sum_{k=0}^{n} a_k (x+1)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}$$

(1)

When the formula is derived, the integrand function $\varphi(x)$ was approximated by an algebraic polynomial of the form $P_n(x) = \sum_{k=0}^{n} a_k (x-a)^k$, that satisfies the condition $(I_{a+}^{\alpha}P_n)(x_j) = (I_{a+}^{\alpha}\varphi)(x_j)$, $j=\overline{0,n}$, where the nodes of the form $x_j = \cos\frac{2j+1}{2n+2}\pi$, $j=\overline{0,n}$ are Chebyshev nodes of the first kind (2).

In order to find the unknowns $\{a_k\}_0^n$, based on the condition $(I_{a+}^{\alpha}P_n)(x_j)=(I_{a+}^{\alpha}\varphi)(x_j)$ $j=\overline{0,n}$, we used the system of linear algebraic equations taking into account $\{a_k\}_0^n$

$$\sum_{k=0}^{n} a_k (x-a)^{k+\alpha}\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}=\frac{1}{\Gamma(\alpha)}\int_a^{x_j}\frac{\varphi(t)}{(x_j-t)^{1-\alpha}}dt$$

$j=\overline{0,n}$, (3)

where the nodes are taken from formula (2). We have proved earlier that this system of linear algebraic equations is uniquely solvable.

Let's calculate $S(I_{-1+}^{\alpha}(\varphi))(x)$. To do this, we substitute the representation (1) and obtain the following:

$$==\sum_{k=0}^{n} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^{k+\alpha}}{t-x}dt.$$

Let's consider the integral $\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^{k+\alpha}}{t-x}dt$ separately. In the numerator we add and subtract $(1+x)^k$. Let's divide the integral into two and use the difference formula of n-th powers $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+...+b^{n-1})$.

Thus, we obtain the following:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^{k+\alpha}}{t-x}dt=\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^k+(1+x)^k-(1+x)^k}{(t+1)^{-\alpha}}\frac{dt}{t-x}=$$

$$=\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\int_{-1}^{1}\frac{(t+1)^s}{(t+1)^{-\alpha}}dt+\frac{1}{\pi}(1+x)^k\int_{-1}^{1}\frac{1}{(t+1)^{-\alpha}}\frac{dt}{t-x}$$

The first integral can be calculated, and the second integral is multiplied and divided by i.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^{k+\alpha}}{t-x}dt=\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\frac{2^{s+\alpha+1}}{s+\alpha+1}+(1+x)^k i\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{(t+1)^{-\alpha}}\frac{dt}{t-x}$$

The integral $\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{(t+1)^{-\alpha}}\frac{dt}{t-x}$ is the integral of Cauchy type. It is known that its main value is calculated by the following formula:

$$V_p\left(\frac{1}{(t+1)^{-\alpha}}\middle|x\right)=\frac{i}{(1+x)^{-\alpha}}\left\{-ctg\pi(-\alpha)+\frac{(1+x)^{-\alpha}}{2^{-\alpha}\pi}\left[\frac{1}{-\alpha}+L\left(\frac{1+x}{2},-\alpha\right)\right]\right\}$$

, where the function $L(z,\alpha)$ is called the quasi-logarithm - these are the functions of a complex variable z, which is defined in a single circle by the series $L(z,\alpha)=\sum_{k=1}^{\infty}\frac{1}{k+\alpha}z^k$, α is a real parameter that satisfies the inequality $-1<\alpha<1$. Thus, we have the following:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(t+1)^{k+\alpha}}{t-x}dt=$$

$$=\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\frac{2^{s+\alpha+1}}{s+\alpha+1}+(1+x)^{k+\alpha}\left\{ctg\pi\alpha+\frac{2^{\alpha}}{(1+x)^{\alpha}\pi}\left[-\frac{1}{\alpha}+L\left(\frac{1+x}{2},-\alpha\right)\right]\right\}$$

And then we obtain that

$$S(I_{-1+}^{\alpha}(\varphi_n))(x)=$$

$$=\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\left(\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\frac{2^{s+\alpha+1}}{s+\alpha+1}+(1+x)^{k+\alpha}\left\{ctg\pi\alpha+\frac{2^{\alpha}}{(1+x)^{\alpha}\pi}\left[-\frac{1}{\alpha}+L\left(\frac{1+x}{2},-\alpha\right)\right]\right\}\right)$$

The final quadrature formula for the calculation of $S(I_{-1+}^{\alpha}(\varphi))(x)$ takes the following form

$$\frac{1}{\pi}\int_a^b\frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt\approx$$

$$\approx\sum_{k=0}^{n}a_k\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\left(\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\frac{2^{s+\alpha+1}}{s+\alpha+1}+(1+x)^{k+\alpha}\left\{ctg\pi\alpha+\frac{2^{\alpha}}{(1+x)^{\alpha}\pi}\left[-\frac{1}{\alpha}+L\left(\frac{1+x}{2},-\alpha\right)\right]\right\}\right)$$

(4)

Let's consider the remainder term of this quadrature formula: $R_n(\varphi)=SI_{-1+}^{\alpha}(\varphi,x)-SI_{-1+}^{\alpha}(P_n,x)$. We take it by the space norm $L_2(-1,1)$. $L_2(-1,1)$. This is the space of measurable functions by $(-1,1)$, that are integrable according to Lebesgue with the degree 2, that is, with the scalar product. The norm of this space is defined by the following formula:

$$\|f\|_{L_2(-1,1)}=\|f\|_2=\left(\int_{-1}^{1}|f(x)|^2dx\right)^{\frac{1}{2}}.$$

Thus, let's consider $\|R_n\varphi\|=\left\|SI_{-1+}^{\alpha}\varphi-SI_{-1+}^{\alpha}P_n\right\|_2$.

Let us derive the operator norm and obtain the following:

$$\|R_n\varphi\|=\left\|SI_{-1+}^{\alpha}\varphi-SI_{-1+}^{\alpha}P_n\right\|_2=\left\|S(I_{-1+}^{\alpha}\varphi-I_{-1+}^{\alpha}P_n)\right\|_2\le\|S\|_{C\to L_2}\left\|I_{-1+}^{\alpha}\varphi-I_{-1+}^{\alpha}P_n\right\|_C.$$

It is known that $\|S\|_{C\to L_2}=1$.

Thus $R_n(\varphi)\le\left\|I_{-1+}^{\alpha}(\varphi,x)-I_{-1+}^{\alpha}(P_n,x)\right\|_C$.

The remaining expression is a special case of an earlier estimate

$$R_n(\varphi) = \frac{((I_{a+}^{\alpha}\varphi)(x))^{(n)}(\xi)}{n!}\omega_n(x).$$

Theorem. In order to calculate $S(I_{-1+}^{\alpha}(\varphi))(x)$ we can use the quadrature formula

$$\frac{1}{\pi}\int_a^b \frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt \approx$$

$$\approx \sum_{k=0}^{n} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\left(\frac{1}{\pi}\sum_{s=0}^{k-1}(1+x)^{k-1-s}\frac{2^{s+\alpha+1}}{s+\alpha+1}+(1+x)^{k+\alpha}\left\{ctg\pi\alpha+\frac{2^{\alpha}}{(1+x)^{\alpha}\pi}\left[-\frac{1}{\alpha}+L\left(\frac{1+x}{2},-\alpha\right)\right]\right\}\right)$$

where $\{a_k\}_0^n$ are determined from the system of linear algebraic equations (3).

For the function $\varphi(x) \in C^{(n)}[a,b]$ the formula has the following error:

$$R_n(\varphi) \le \frac{((I_{-1+}^{\alpha}\varphi(x))^{(n)}(\xi)}{n!}\omega_n(x).$$

Then we checked the obtained results in the Wolfram Mathematica system.

The calculation results $\frac{1}{\pi}\int_{-1}^{1} \frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt$ using the formula (4) for the function $\varphi(t) = t^2$ at $\alpha = 0,5$ and given values x are presented in Table 1.

Table 1.

x	-0,9	-0,6	-0,3	0	0,3	0,6	0,9
$S(I_{-1+}^{\alpha}(\varphi))(x)$	1,01032	0,52291	0,26029	0,18105	0,22994	0,31759	0,17737

The calculation results $\frac{1}{\pi}\int_{-1}^{1} \frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt$ using the formula (4) for the function $\varphi(t) = t^3$ at $\alpha = 0,5$ and given values x are presented in Table 2.

Table 2.

x	-0,9	-0,6	-0,3	0	0,3	0,6	0,9
$S(I_{-1+}^{\alpha}(\varphi))(x)$	-0,81854	-0,28423	-0,05507	0,04391	0,14319	0,30419	0,39842

We compared the values of the integral calculated by the formula (4) and by the formula $S(I_{-1+}^{\alpha}(\varphi))(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt$, using the functions of the Wolfram Mathematica system (Table 3).

Table 3.

x	(4)	$\frac{1}{\pi}\int_{-1}^{1} \frac{(I_{-1+}^{\alpha}\varphi)(t)}{t-x}dt$
-0,5	0,41225	0,39446
-0,4	0,32521	0,35487
-0,3	0,26029	0,31561
-0,2	0,21584	0,27951
-0,1	0,19009	0,24886
0	0,18105	0,22568
0,1	0,18647	0,21188
0,2	0,20379	0,20938
0,3	0,22994	0,22029

0,4	0,26119	0,24706
0,5	0,29267	0,29281

The maximum values of value difference modulus calculated by different formulas make 0.06367.

Conclusions

The developed quadrature formula allows us to calculate the singular integral with the Cauchy kernel from the fractional Riemann-Liouville integral.

Summary

Thus, they derived the quadrature formula for the computation of $S(I^{\alpha}_{-1+}(\varphi))(x)$. The error of the following formula is estimated: $R_n(\varphi) \leq \frac{((I^{\alpha}_{-1+}\varphi(x))^{(n)}(\xi)}{n!}\omega_n(x)$. The necessary calculations are made in the Wolfram Mathematica system.

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