Artículo de investigación

Sigma model realization in Spinor theory

Realización del modelo sigma en la teoría de Spinor

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Abstract

16-spinor field is suggested to unify Skyrme and Faddeev models (Sigma Model) as topological soliton. The model is based on the well-known 8-spinor discovered by Italian geometer Brioschi. Study the structure of topological charge in the lepton sector, lepton charge is identified with the Hopf index. Estimating the energy, mass and spin of the model. Two models Skyrme and Faddeev were unifying by using 16-spinor field. In this paper we found solution at small distance behavior we can develop model and study behavior at large distance and match between two cases.

Keywords: 16-spinor, Sigma model, Homotopy groups.

Resumen

Se sugiere un campo de 16 espines para unificar los modelos Skyrme y Faddeev (Modelo Sigma) como solitón topológico. El modelo se basa en el conocido 8-spinor descubierto por el geómetra italiano Brioschi. Se estudia la estructura de la carga topológica en el sector de leptones, la carga de leptones se identifica con el índice Hopf. Estimación de la energía, masa y giro del modelo. Dos modelos Skyrme y Faddeev se unificaron utilizando un campo de 16 espines. En este artículo encontramos una solución a un comportamiento a pequeña distancia, podemos desarrollar modelos y estudiar el comportamiento a grandes distancias y hacer coincidir dos casos.

Palabras clave: 16 spinor, modelo Sigma, grupos de homotopía.

Introduction

We started with two models first one is Skyrme model [1] Which is an effective low-energy action of QCD (Quantum Chromodynamics) theory and study the internal structure of hadrons. The baryon number (B) is identified with topological charge $Q = deg(S^3 \to S^3)$ and serving as generator of Homotopy group ($\pi_3(S^3) = \mathbb{Z}$). The second one was announced by Faddeev in 1972 using the same idea to describe leptons. leptons are consider as topological soliton endowed with the Hopf invaiant Q_H, the latter one playing the role of the lepton number L= Q_H , serving as generator of the homotopy group $\pi_3(S^2) = \mathbb{Z}$. to unify Faddeev- Skyrme models, we introduce 16-spinor filed: $\Psi = \Psi_1 \oplus \Psi_2$, $\lambda_i = I_4 \otimes \sigma_i \otimes I_2$, Ψ_i , $i = 1,2$, Ψ_i being 8-spinor and σ_i stands for Pauli matrices for each 8-spinor Ψ_i the following Brioschi identity form (2013):

$$j_\mu j^\mu - \widetilde{j_\mu} \widetilde{j^\mu} = s^2 + p^2 + \vec{v}^2 + \vec{a}^2 \qquad (1)$$

Starting with bilinear spinor quantities:

$$s = \overline{\Psi}\Psi , \quad p = i\overline{\Psi}\gamma_5\Psi , \quad \vec{v} = \overline{\Psi}\lambda\Psi, \quad \vec{a} = i\overline{\Psi}\gamma_5\lambda\Psi$$
$$j_\mu = \overline{\Psi}\gamma_\mu\Psi , \quad \widetilde{j_\mu} = \overline{\Psi}\gamma_\mu\gamma_5\Psi$$

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with γ_{μ} being Dirac matrices and $\overline{\Psi} = \Psi^{+}\gamma_{0}$. In view of the Brioschi identity in general we can use Higgs potential V implying the role of spontaneous symmetry breaking:

$$V(j_{\mu}j^{\mu}) = \frac{\sigma^2}{8}(j_{\mu}j^{\mu} - \varkappa_0^2)^2 \qquad (2)$$

where σ and $\varkappa_o$ are some constant parameters of our mode. To localized soliton as configurations we use the boundary condition at space infinity:

$$\lim_{r\to\infty} j_{\mu}j^{\mu} = \varkappa_0^2 \qquad (3)$$

There are two kinds of topological solitons are possible depend on your choice of manifold ($s^3 or\ s^2$) as phase spaces. in this work we are going to talk about lepton sector, so s^2 field manifold is determining by the O(3)-invariant V^2 and corresponds to Faddeev model. Introducing the 16-spinor of the form: $\{\Psi = \oplus_{i=1}^{2}(\varphi_i \oplus \chi_i \oplus \xi_i \oplus \theta_i)\}$ where $\varphi_i, \chi_i, \xi_i, \theta_i$ being two spinors, with the mirror symmetry of lepton sector we start:

$$\Psi \to \gamma_0 \Psi \qquad (4)$$

One finds from invariance condition (4) that $\varphi_i = \chi_i$, $\xi_i = \theta_i$ to get an effective 16-spinor with the Brioschi identity:

$$\frac{j_{\mu}j^{\mu} - V^2}{16} = (|\varphi_1|^2 + |\xi_1|^2)(|\varphi_2|^2 + |\xi_2|^2) - |\varphi_1^{+}\varphi_2 + \xi_1^{+}\xi_2|^2 \geq 0 \qquad (5)$$

1. The effective nonlinear 16-spinor field model

Let us consider the following Lagrangian density for nonlinear spinor model which introduce in (Yu. P. Rybakov, 2011):

$$\mathcal{L}_{spin} = \frac{1}{2\lambda^2}\overline{\partial_{\mu}\Psi}\gamma^{v}j_{v}\partial^{\mu}\Psi + \frac{\epsilon^2}{4}f_{\mu v}f^{\mu v} - \frac{\sigma^2}{8}(j_{\mu}j^{\mu} - \varkappa_0^2)^2 \qquad (6)$$

where $f_{\mu v}$ stands for the anti-symmetric tensor of Sigma model:

$$f_{\mu v} = (\Psi^{+}\partial_{\mu}\Psi)(\partial_{v}\Psi^{+}\Psi) - (\Psi^{+}\partial_{v}\Psi)(\partial_{\mu}\Psi^{+}\Psi) \qquad (7)$$

Starting with λ and ϵ are constants parameters of the sigma model and the first term was given in Lagrangian density (6) generalized the sigma model and includes the projector on positive energy states can be written in the form: ($p = \gamma^0\gamma^v j_v$), the second term in (6) gives the generalization of Faddeev or Skyrme term, the model admits this discrete symmetry $\varphi_i \leftrightarrow \chi_i$ and $\xi_i \leftrightarrow \theta_i$ to identify $\varphi_i = \chi_i$ also $\xi_i = \theta_i$,let us start with static axially-symmetric configurations assume that spinors are eigenvectors of angular momentum operator $J_3 = -i\partial_{\Phi} + \frac{1}{2}\sigma_3$, has a close similarity to rotations around the z-axis:

$$\begin{pmatrix} J_3\varphi_1 = \frac{1}{2}\varphi_1, & J_3\varphi_2 = -\frac{1}{2}\varphi_1 \\ J_3\,\xi_1 = \frac{1}{2}\,\xi_1, & J_3\,\xi_2 = -\frac{1}{2}\,\xi_2 \end{pmatrix} \qquad (8)$$

where Φ is the azimuth angle and σ_3 is Pauil matrix, by using isotropic spherical coordinates:

$$x^1 = \ln(r/r_0)\ , \quad x^2 = \theta, \quad x^3 = \Phi \qquad (9)$$

where r_0 is a scale parameter, consider the metric in isotropic spherical coordinates:

$$ds^2 = dt^2 - e^{2x}(dx^2 + d\theta^2 + sin^2\theta d\Phi^2) \qquad (10)$$

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we have found the following structure of solutions to equation (8):

$$\begin{pmatrix} \varphi_1 = col(a_1, b_1 \exp[i\Phi]) \\ \varphi_2 = col(a_2 \exp[-i\Phi], b_2) \\ \xi_1 = col(c_1, d_1 \exp[i\Phi]) \\ \xi_2 = col(c_2, d_2 \exp[-i\Phi]) \end{pmatrix} \quad (11)$$

Note that all functions (a_i, b_i, c_i, d_i) depend on two coordinates (x, θ) the structure of Lagrangian in (6) admits the following discrete sysmmetry group:

$$\Psi_i \to \sigma_3 \otimes \sigma_3 \Psi^*{}_i \quad (12)$$

The discrete symmetry above show us that we can introduce these functions a_i, b_i, c_i, d_i, as power series in $(i\cos\theta)$ with the additional multiplication of (b_i, d_i) by $(i\sin\theta)$, to get the structure of our model filed configuration, let start with the contribution of only the first two terms:

$$\begin{pmatrix} a_1 = a + i\,f\cos\theta, & b_1 = if\sin\theta \\ a_2 = b + i\,g\cos\theta, & b_2 = ig\sin\theta \\ c_1 = c + i\,h\cos\theta\,, & d_1 = ih\sin\theta \\ c_2 = d + i\,k\cos\theta\,, & d_2 = ik\sin\theta \end{pmatrix} \quad (13)$$

where (a, f, b, g, c, h, d, k) are some real functions, substituting (13) in (6) we obtain the following Lagrangian density:

$$\mathcal{L}_{spin} = \frac{-e^{-2x}}{2\lambda^2}[a^2+f^2+b^2+g^2+c^2+h^2+d^2+k^2] * [(\partial a)^2+(\partial f)^2+(\partial b)^2+(\partial g)^2+(\partial c)^2+ (\partial h)^2+(\partial d)^2+(\partial k)^2+2(b^2+g^2+h^2+k^2]+4\epsilon^2 e^{-4x}[(af+bg+ch+dk)^2+ (f^2-g^2+h^2-k^2)^2]*[a\,\partial a+f\partial f+b\partial b+g\partial g+c\partial c+h\partial h+d\partial d+k\partial k]-\sigma^2 e^{3x}[(a^2+f^2+ b^2+g^2+c^2+h^2+d^2+k^2)^2-\ \varkappa_0^2]^2 \quad (14)$$

To find the structure of the energy functional E , let us define these functions (a, f, b, g, c, h, d, k) in new variables:

$$\begin{aligned} a &= R^{\frac{1}{2}} cos\ominus\cos(\xi+\eta), & f &= R^{\frac{1}{2}} sin\ominus\cos(\xi-\eta), \\ b &= R^{\frac{1}{2}} cos\ominus\sin(\xi+\eta), & g &= R^{\frac{1}{2}} sin\ominus\sin(\xi-\eta), \\ c &= R^{\frac{1}{2}} cos\ominus\cos(\xi-\eta), & h &= R^{\frac{1}{2}} sin\ominus\cos(\xi+\eta), \\ d &= R^{\frac{1}{2}} cos\ominus\sin(\xi-\eta), & k &= R^{\frac{1}{2}} sin\ominus\sin(\xi+\eta), \end{aligned}$$

one easily finds the energy functional E

$$E = \int_{-\infty}^{\infty} dx\ \left\{\frac{e^x}{2\lambda^2}\left[\frac{1}{2}\mathrm{R}^{/2}+2R^2(\ominus^{/2}+\xi^{/2}+\eta^{/2}+2sin^2\ominus)\right]+8\epsilon^2 e^{-x}\mathrm{R}^{/2}R^2 sin^2\ominus\,(sin^2 \ominus\, cos^2 2(\xi-\eta)+\ cos^2\ominus * cos^2\, 2(\xi+\eta)+\sigma^2 e^{3x}(R^2-\frac{\varkappa_0^2}{4})^{\ 2}\right\}$$

(15)

2. Lepton charge

To find lepton charge L we need to calculate the Whitehead integral (Whitehead J.H.C., 1947; Whitney H., 1957; D. Husemoller, 1966).

$$Q_H = \frac{1}{16\pi^2}\int d^3x\,(b\;rot\;b) \quad (16)$$

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We defined the vector b by the relation:($\partial_i b_k - \partial_k b_i = \epsilon^{abc}\partial_i n^a \partial_k n^b n^c$), By definition the auxiliary 2-spinor take this form:

$$\chi \ = \mathrm{col}(\cos \mathrm{A} + i \sin \mathrm{A} \ \cos \mathrm{B} \ , \sin \mathrm{A} \ \sin \mathrm{B} \ \mathrm{e}^{iC}) \qquad (17)$$

where $(\mathrm{A}, \mathrm{B}, C)$ are Angular coordinates on s^2, we also have the following relations for vector (b):

$$\begin{Bmatrix} rot\ b = -2i[\nabla\chi^{+}\nabla\chi\ ] \\ \mathrm{b} = \mathrm{lm}(\chi^{+}\nabla\chi) \\ n = (\chi^{+}\lambda\,\chi) \end{Bmatrix} \qquad (18)$$

Inserting (18) into (16) to get:

$$Q_H = \frac{1}{4\pi^2}\int d^3x\ \sin^2 \mathrm{A} \sin \mathrm{B}\ ([\nabla\mathrm{A}\nabla\mathrm{B}]\nabla\mathrm{C}\ ) \qquad (19)$$

new variables can read ($\sin \mathrm{A}\ \sin \mathrm{B} = \sin\ \frac{\beta}{2}$, $\tan \mathrm{A} \cos \mathrm{B}\ \tan\rho$) Therefore we obtain:

$$Q_H = \frac{-1}{8\pi^2}\int d^3x\ \sin\beta\ ([\nabla\beta\nabla\rho]\nabla\mathrm{C}\ ) \qquad (20)$$

One can calculate the structure of vector $V = \overline{\Psi}\lambda\Psi$ which is determining S^2 - manifold for lepton sector:

$$V_1 = 4Re(\varphi_1^{+}\varphi_2 + \xi_1^{+}\xi_2), \qquad V_2 = 4Im(\varphi_1^{+}\varphi_2 + \xi_1^{+}\xi_2)$$
$$V_3 = 2(|\varphi_1|^2 - |\varphi_2|^2 + |\xi_1|^2 - |\xi_2|^2)$$

The Angular variables $\rho\ and\ \beta$ are define for the unit vector n: ($n_1 = \sin\beta\ \cos\gamma, n_2 = \sin\beta\ \sin\gamma,\ n_3 = \cos\beta$), also Angular variables can be expressed like ($n_1 + in_2 = \sin\beta\ \mathrm{e}^{i\gamma}$) one finds this relation:

$$\gamma = C - \ \rho \qquad (21)$$

Now let us introduce new angular coordinate μ it can be found from (11) and the definition of the vector: $\gamma = \mu(r,\beta) - \emptyset$, One gets:

$$Q_H = \frac{1}{8\pi^2}\int d^3x\ \sin\beta\ ([\nabla\beta\nabla\mu]\nabla\emptyset\ ) \qquad (22)$$

Using the definition of the vector above and (22), one obtains solution of lepton charge:

$$Q_H = \frac{1}{4\pi}\int_0^{r_0} dr\ [\mu] sin\beta$$
$$Q_H = n \qquad (23)$$

notice that boundary conditions:

$$\begin{pmatrix} cos\beta(r=r_0,\theta=\pi/2)=1 \\ cos\beta(r=0,\theta=\pi/2)=-1 \end{pmatrix} \qquad (24)$$

where $[\mu] = -2\pi n$, the second term of energy functional:

$$8\epsilon^2 e^{-x}\mathrm{R}^{/2}R^2 sin^2\ominus\ (sin^2\ominus\ cos^2 2(\xi-\eta) + cos^2\ominus\ cos^2\ 2(\xi+\eta)$$

we may apply the conditions (24) if we choose approximately $\xi = 0, \eta = \pi/2$.Then we get the energy functional of the form:

$$E = \int_{-\infty}^{\infty} dx \frac{e^x}{2\lambda^2}(\frac{1}{2}\mathrm{R}^{/2} + 2R^2(\ominus^{/2} + \xi^{/2} + \eta^{/2} + 2sin^2\ominus) + 8\epsilon^2 e^{-x}\mathrm{R}^{/2}R^2 sin^2\ominus + \sigma^2 e^{3x}(R^2 - \frac{\varkappa_0^2}{4})^{\,2}$$

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To estimate the energy function, we introduce trial functions:

$$\{R = A\,tanh(\alpha x) + B,\ \ominus = 2tan^{-1}(e^{-\alpha x})\} \qquad (25)$$

with the boundary conditions: $\left\{R(+\infty) = \frac{\varkappa_0}{2},\ R(-\infty) < \frac{\varkappa_0}{2}, \ominus(-\infty) = \pi,\ \ominus(+\infty) = 0,\quad \ominus(0) = \pi/2\right\}$. one can obtain:

$$E = \int_{-\infty}^{\infty} dx \frac{r_0 e^x}{2\lambda^2}\left(\frac{1}{2}A^2\alpha^2 sech^4(\alpha x)\right) + (A\,tanh(\alpha x) + B)^2\left(\frac{2\alpha^2}{cosh^2(\alpha x)} + \frac{2\alpha}{\cosh(\alpha x)}\right) + (8\epsilon^2 {r_0}^{-1} e^{-x} A^2\alpha^2 sech^6(\alpha x)(A\,tanh(\alpha x) + B)^2) + ({r_0}^3\sigma^2 e^{3x}((A\,tanh(\alpha x) + B)^2 - \frac{\varkappa_0^2}{4})^2)$$

The integrals above can be evaluated by using Numerical Methods:

$$E = \frac{2r_0\alpha}{\lambda^2}(A^2 + 3B^2) + \frac{128A^2\alpha\epsilon^2}{15r_0}\left(\frac{A^2}{7} + B^2\right) + \frac{4\sigma^2{r_0}^3A^2}{3\alpha}\left(\frac{A^2}{5} + (A + 2B)^2\right)$$

Let us denote that $A + B = \varkappa_0/2$ and for each function approximate value:

$$A \approx 0.222\varkappa_0 \quad , B \approx 0.278\varkappa_0$$

3. Spin and mass of soliton

Let us now calculate the spin value of the soliton. the structure of Lagrangian was given in (6) for the configurations with nontrivial spin value one can includes the interaction function with the gravitational field or electromagnetic field. for Perturbation theory in the electromagnetic field it is the standard way to do it, let us first introduce the extended derivative of the spinor field:

$$D_\mu = \partial_\mu - ie_0 A_\mu \varGamma_e \qquad (26)$$

Where A_μ is the electromagnetic potential and $\varGamma_e$ electric charge operator, with e_0 being the electromagnetic coupling constant, by using the definition of the spin density:

$$\rho_s = 2Re\left[\frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)} J_3\Psi\right] \qquad (27)$$

One finds from (26) that:

$$\rho_s = \frac{e_0}{\lambda^2} j_0 A_0(\Psi^+ \varGamma_e J_3 \Psi) \qquad (28)$$

where $\varGamma_e$ the charge operator such that:

$$\varGamma_e = \frac{1}{2}(\lambda_3 - 1) \qquad (29)$$

If one compares between (6) and (28) that leading term: $(\Psi^+\varGamma_e J_3\Psi) = (b^2 + g^2 + d^2 + k^2)$, so the spin value is given by the integral:

$$S = \frac{8\pi e_0 {r_0}^3}{\lambda^2}\int_{-\infty}^{+\infty} dx e^{3x} R A_0 (b^2 + g^2 + d^2 + k^2) \qquad (30)$$

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In view of the trial functions: $(\xi + \eta = 2\arctan(sinh\alpha x))$ In the first approximation one can put $\xi \approx 0$ then we find:

$$f^2 + g^2 + c^2 + d^2 \approx 2Rsin^2(\eta) = 8R\frac{tanh^2(\alpha x)}{cosh^2(\alpha x)}$$

For the scalar potential A_0 let us use the solution found in [8]:

$$A_0 = \frac{e_0}{r_0(1+e^{2x})^{1/2}} \qquad (31)$$

with boundary condition $R^2(+\infty) = \varkappa_0^2/4$ one finds:

$$s = \frac{38.4\,\pi e_0 \varkappa_0^2 {r_0}^3}{\lambda^2} \qquad (32)$$

Value of spin is $(s = 1/2)$ in natural units $(h/2\,\pi = 1)$. finally one can obtain the mass of the particle-soliton by using this relation:

$$m = 16\pi\left\{\frac{2r_0\alpha}{\lambda^2}(A^2 + 3B^2) + \frac{128A^2\alpha\epsilon^2}{15r_0}\left(\frac{A^2}{7} + B^2\right) + \frac{4\sigma^2{r_0}^3A^2}{3\alpha}\left(\frac{A^2}{5} + (A + 2B)^2\right)\right\}$$

Conclusion

Study the structure of topological charge in the lepton sector, lepton charge is identified with the Hopf index. Estimating the energy, mass and spin of the model. Two models Skyrme and Faddeev were unifying by using 16-spinor field. In this paper we found solution at small distance behavior we can develop model and study behavior at large distance and match between two cases.

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